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Research Article

A mathematical model on thermally induced vibration of tapered rectangular plate

Narinder Kaur[1], Anupam Khanna[2,*]

[1] Department of Mathematics, Statistics and Physics, Punjab Agricultural University, Ludhiana, India
[2] Department of Mathematics, D.A.V. College, Sadhaura, Yamunanagar, Haryana, India
*e-mail: rajieanupam@gmail.com

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Abstract: A theoretical mathematical model on vibration of rectangular plate is discussed. In this study, the vibration of the bi-parabolic tapered rectangular plate is analyzed under two different boundary conditions i.e. clamped (C-C-C-C) and simply supported (SS-SS-SS-SS). Also, the author considered bi-parabolic variation in the temperature field which occurs due to thermally induced vibration in rectangular plate. Results of frequency for the first two modes of vibration are obtained by using Rayleigh-Ritz method. Variations in frequency for first two modes of vibration at different values of structural parameters (thermal gradient, taper constants, and aspect ratio) and boundary conditions are well explained with the help of graphs.

Keywords: vibration; tapered; thermal gradient; frequency; structural parameters; aspect ratio

I. INTRODUCTION

Tapered rectangular plates made-up of composite materials are vigorously used to prevent uncontrolled vibration of structures and machines. These materials not only control excessive and unpleasant stresses of thermally induced vibration but also provide high strength, low cost, and reliability to the structure.

Research in the field of vibration of plates is continuously motivating scientists and engineers to develop more realistic theoretical mathematical models for future aspects. A lot of theoretical and practical research work in the form of literature is already available in this field but still pre-information about first few modes of the frequency of vibration are always required to make more efficient, reliable, and authentic structures, especially; when structure work under the influence of temperature.

Linear free vibrations analysis of rectangular plate resting on translational and rotational supports at all edges is performed by Babahammou and Benamar [2]. Behera and Chakarvarty [3] used Boundary characteristic orthogonal polynomials as shape functions in the Rayleigh–Ritz method for static analysis of nanobeams. This method can handle any set of classical boundary conditions (viz., clamped, simply supported and free) with ease. The free-flexural vibration analysis of circular plates over point supports, partial internal curved supports, and with mixed-edge boundary conditions is analyzed by Liew [4]. Wang [5] investigated vibration analysis and optimization of a rectangular plate with a flanging hyperellipse cutout, numerically. In this study, finite element method (FEM) is applied to perform parametric studies on various plates in different boundary conditions. Farsani et.al. [6] analyzed the free vibrations of rectangular sandwich plates with compressible core bu using extended higher-order sandwich plate theory. In this paper, effects of the plate side-to-thickness ratio, in-plane aspect ratio, and core-to-face sheets thickness ratio on the natural frequencies are discussed. Since the finite element analysis of deformations for plates made of functionally graded materials cannot be carried out with the use of commercial FE packages, a consistent method of analytical analysis is proposed by Aleksander Muc [7]. The effect of the non-homogeneity present in plate's material of orthotropic rectangular plate in the form of density is explained by Lal [8]. Buckling analysis of rectangular plates having two opposite edges (x=0 & x=a) simply supported and others two edges (y=0 and y=b) may be clamped, simply supported or free is formulated by Kang and Leissa [9]. Gupta et.al. [12] analyzed forced vibrations of linearly tapered rectangular plate with non-homogeneity present in density of the plate's material. Results for simply supported-free-simply supported- free boundary condition are obtained and shown in graphical form. Khanna and Kaur [14] obtained the frequency of visco-elastic tapered non-homogeneous rectangular plate under bi-parabolic thermal condition by using Rayleigh-Ritz technique. Kaur [17] employed Rayleigh-Ritz method to discuss vibrations of different triangular plates having fully clamped boundary with bi-linear tapering in thickness.

By using classic Rayleigh-Ritz method, results are calculated for various values of plate parameters i.e. thermal gradient, taper constants, and aspect ratio with respect to two boundary conditions (Clamped and Simply Supported). Vibrational behavior of frequency for both the modes is explained with the help of graphs. Developments of this model are written in the form of conclusions.

II. MATERIALS AND METHODS

1. Geometry of the plate

Rectangular plate with bi-parabolic tapering in thickness along X- & Y- axis is shown in **Fig. 1**. The plate lies along the X-Y plane with the Z-axis representing the thickness. Here OC=H_0. The parabolic tapering effect creates a smooth curvature with a gradual increase in thickness as one moves from point C towards point B or F.

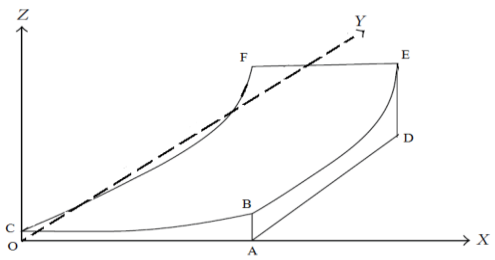

Figure 1. *Rectangular Plate with Bi-parabolic Tapering*

2. Analysis of Motion

Fourth-order differential equation of motion for tapered rectangular plate in Cartesian coordinate is [15]:

$$\begin{bmatrix} D_1\left(\frac{\partial^4 W}{\partial x^4}+2\frac{\partial^4 W}{\partial x^2\partial y^2}+\frac{\partial^4 W}{\partial y^4}\right)+2\frac{\partial D_1}{\partial x}\left(\frac{\partial^3 W}{\partial x^3}+2\frac{\partial^3 W}{\partial x\partial y^2}\right)+ \\ 2\frac{\partial D_1}{\partial y}\left(\frac{\partial^3 W}{\partial y^3}+2\frac{\partial^3 W}{\partial y\partial x^2}\right)+\frac{\partial^2 D_1}{\partial x^2}\left(\frac{\partial^2 W}{\partial x^2}+\upsilon\frac{\partial^2 W}{\partial y^2}\right)+ \\ \frac{\partial^2 D_1}{\partial y^2}\left(\frac{\partial^2 W}{\partial y^2}+\upsilon\frac{\partial^2 W}{\partial x^2}\right)+ \\ 2(1-\upsilon)\frac{\partial^2 D_1}{\partial x\partial y}\frac{\partial^2 W}{\partial x\partial y} \end{bmatrix}-\rho p^2 H W=0 \quad (1)$$

where, D_1 is the flexural rigidity of the plate's material, $W = W(x, y)$ is the deflection function, ν is poisson ratio, ρ is mass per unit volume, and H is the thickness of the plate.

Unwanted vibrations in the plate generate a thermal effect within the structure, which directly alters the material's vibrational properties. In the present study, the authors assumed bi-parabolic variation in thermal effect as [13]:

$$\tau=\tau_0\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{y^2}{b^2}\right) \quad (2)$$

where τ denotes the temperature excess above the reference temperature at any point on the plate and τ_0 denotes the temperature at any point on the boundary of plate.

Relation between modulus of elasticity (Y) with temperature variation in plate is expressed as follows [1]:

$$Y=Y_o(1-\gamma\tau) \quad (3)$$

where Y_0 is the value of the Young's modulus at reference temperature i.e. $\tau = 0$ and γ is the slope of variation of Y and τ. On using equation (2) in equation (3), one obtains:

$$Y=Y_0\left(1-\alpha\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{y^2}{b^2}\right)\right) \quad (4)$$

where, $\alpha=\gamma\tau\,(0\le\alpha\le 1)$ is thermal gradient.

Tapering in the thickness of the rectangular plate is considered bi-parabolically in the present study as [10]:

$$H=H_0\left(1+\beta_1\frac{x^2}{a^2}\right)\left(1+\beta_2\frac{y^2}{b^2}\right) \quad (5)$$

Here a & b are the dimensions of the rectangular plate and β_1 & β_2 are taper constants in x-direction and y-direction respectively. Taper constants range from 0 to 1 i.e. $0\le\beta_1\le 1$ & $0\le\beta_2\le 1$. Also, H_0 is the thickness of the plate at $x = y = 0$.

Flexural rigidity of any material indicates the bending stiffness of the material. In other words, the flexural rigidity shows 'how much a plate is rigid against bending'. In this study, flexural rigidity is considered as [10]:

$$D_1=\frac{Y\times H^3}{12(1-\nu^2)} \quad (6)$$

Using the values of Y and H from equations (4) and (5), the flexural rigidity becomes:

$$D_1=\frac{Y_0\left[1-\alpha\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{y^2}{b^2}\right)\right]\times H_0^3\left[\left(1+\beta_1\frac{x^2}{a^2}\right)^3\left(1+\beta_2\frac{y^2}{b^2}\right)^3\right]}{12(1-\nu^2)} \quad (7)$$

3. **Solution of Frequency Equation**

To solve the frequency equation, the authors applied Rayleigh-Ritz technique which states that maximum strain energy (S_E) and maximum kinetic energy (K_E) must be taken equally. So the following consideration is mandatory for solving the problem [11]:

$$\delta(S_E - K_E) = 0 \tag{8}$$

The mechanical properties of the vibrating structures are directly affected by the boundary of the structure. Therefore, it is necessary to analyze the effect of boundary conditions on the vibration of the plate. In this paper, two different boundary conditions are studied and their effects on the frequency of the vibrating plates are examined by using the latest computational technique.

Two term deflection function for clamped boundary (9) and simply supported boundary (10) is taken as follows [8]:

$$W = [(\frac{x}{a})(\frac{y}{b})(1-\frac{x}{a})(1-\frac{y}{b})]^2 \times [A_1 + A_2(\frac{x}{a})(\frac{y}{b})(1-\frac{x}{a})(1-\frac{y}{b})] \tag{9}$$

$$W = [(\frac{x}{a})(\frac{y}{b})(1-\frac{x}{a})(1-\frac{y}{b})] \times [A_1 + A_2(\frac{x}{a})(\frac{y}{b})(1-\frac{x}{a})(1-\frac{y}{b})] \tag{10}$$

where A_1 and A_2 are arbitrary constants.

Now, assuming the non-dimensional variables as [15]:

$$X = \frac{x}{a},\ Z = \frac{y}{a},\ \overline{W} = \frac{W}{a},\ \overline{H} = \frac{H}{a} \tag{11}$$

The expressions for kinetic energy (K_E) and strain energy (S_E) are [13]:

$$K_E = \frac{1}{2}\rho p^2 \overline{H}_0 a^5 \int_0^1 \int_0^{b/a} [(1+\beta_1 X^2)(1+\beta_2 \frac{a^2 Z^2}{b^2})\overline{W}^2] dZdX \tag{12}$$

$$S_E = \frac{Y_0}{24(1-\upsilon^2)} \overline{H}_0^3 a^3 \int_0^1 \int_0^{b/a} [(1+\beta_1 X^2)^3 (1+\beta_2 \frac{a^2 Z^2}{b^2})^3 \times [(\frac{\partial^2 \overline{W}}{\partial X^2})^2 + (\frac{\partial^2 \overline{W}}{\partial Z^2})^2 + 2\upsilon(\frac{\partial^2 \overline{W}}{\partial X^2} \times \frac{\partial^2 \overline{W}}{\partial Z^2}) + 2(1-\upsilon)(\frac{\partial^2 \overline{W}}{\partial X \partial Y})^2] dZdX \tag{13}$$

Using equations (12) and (13) in equation (8), one gets

$$\delta(S_{E_1} - \lambda^2 K_{E_1}) = 0 \tag{14}$$

where

$$K_{E_1} = \int_0^1 \int_0^{b/a} [(1+\beta_1 X^2)(1+\beta_2 \frac{a^2 Z^2}{b^2})\overline{W}^2] dZdX \tag{15}$$

$$S_{E_1} = \int_0^1 \int_0^{b/a} [(1+\beta_1 X^2)^3 (1+\beta_2 \frac{a^2 Z^2}{b^2})^3 \times [(\frac{\partial^2 \overline{W}}{\partial X^2})^2 + (\frac{\partial^2 \overline{W}}{\partial Z^2})^2 + 2\upsilon(\frac{\partial^2 \overline{W}}{\partial X^2} \times \frac{\partial^2 \overline{W}}{\partial Z^2}) + 2(1-\upsilon)(\frac{\partial^2 \overline{W}}{\partial X \partial Z})^2] dZdX \tag{16}$$

Here, $\lambda^2 = \frac{12\rho p^2}{Y_0 H_0^2}$ is a frequency parameter.

Equation (14) consists of two unknown constants i.e. A_1 and A_2 arising due to the substitution of W (for both Models). These constants can be obtained easily as [10]:

$$\frac{\partial}{\partial A_n}(S_{E_1} - \lambda^2 K_{E_1}) = 0,\ n = 1,2 \tag{17}$$

From equation (15), the author obtains

$$b_{n1}A_1 + b_{n2}A_2 = 0,\ n = 1,2 \tag{18}$$

where b_{n1} and b_{n2} involve plate's parameters.

For a non-trivial solution, determinant of the coefficient of equation (18) must be zero. So, one gets the frequency equation as [16]:

$$\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} = 0 \tag{19}$$

With the help of equation (19), one can obtain a quadratic equation in λ^2 from which the two values of frequency parameter for both the modes of vibration can be evaluated easily.

III. RESULTS AND DISCUSSION

First two modes of frequency are obtained for both mathematical models for different values of the structural parameters i.e. thermal gradient, taper constants, and aspect ratio of rectangular plate. Numerical findings are computed for an alloy "Duralium" which is frequently used in the making of parts of ships, submarines etc. Due to an alloy of aluminium and copper, duralumin becomes as strong as steel along with light weight and cheap in cost. Here poisson ratio is considered as constant i.e. 0.345 and thickness of the plate at $X = Y = 0$ is taken as 0.01 M.

Variations in the frequency for both the modes of vibration are cautiously observed and elaborately presented in **Fig. 2** to **Fig. 13** for both boundary conditions as follows.

In **Fig 2** and **3**, variation in the first mode of the frequency for zero thermal effect is plotted with respect to increasing values of the taper constant along x-direction (β_1) at different values of the second taper constant along y-direction (β_2) for both boundary conditions respectively. Similarly, in **Fig. 4** and **5**, second mode of the frequency with respect to increasing values of β_1 at different values of β_2 are shown for both boundary conditions at α =0.0. From **Fig. 2** to **5**, a continuous increment in both the modes of frequency for both boundary conditions is observed with respect to varying taper constants. Authors also noticed that first mode of frequency for simply supported boundary condition is slightly greater than the frequency for clamped boundary condition at corresponding values of structural parameters. It is interesting to see that mode 2 for simply supported boundary condition is quite lesser than the mode 2 for clamped boundary at corresponding values of taper constants.

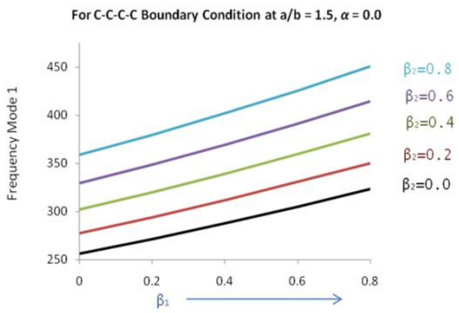

Figure 2. *Frequency (Mode 1) Vs β_1 & β_2 at α=0.0 , a/b=1.5 for clamped boundary*

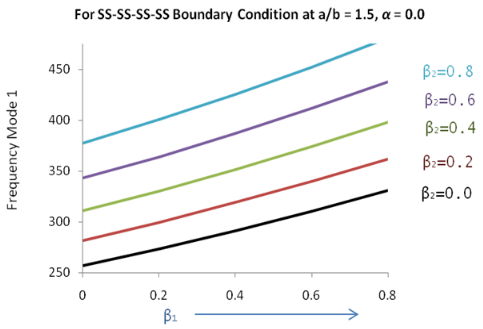

Figure 3. *Frequency (Mode 1) Vs β_1 & β_2 at α=0.0 , a/b=1.5 for simply supported boundary*

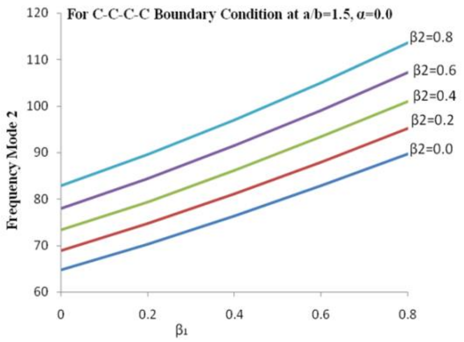

Figure 4. *Frequency (Mode 2) Vs β_1 & β_2 at α=0.0, a/b=1.5 for clamped boundary*

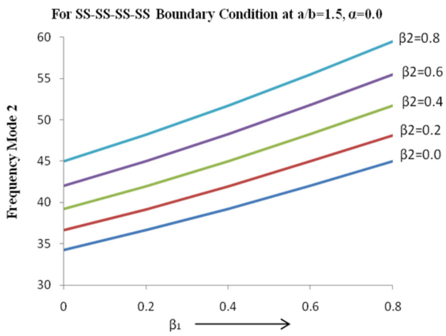

Figure 5. *Frequency (Mode 2) Vs β_1 & β_2 at α=0.0, a/b=1.5 for simply supported boundary*

In **Fig. 6** to **9**, variations in frequency for both the modes of vibration are presented at non-zero thermal gradient i.e. α = 0.6 for various values of taper constants and fixed aspect ratio (=1.5). Surprisingly both the modes of frequency are reported lesser in **Fig. 6** to **9** as compared to **Fig. 2** to **5** for corresponding values of structural parameters while both the modes increase with increasing values of taper constants.

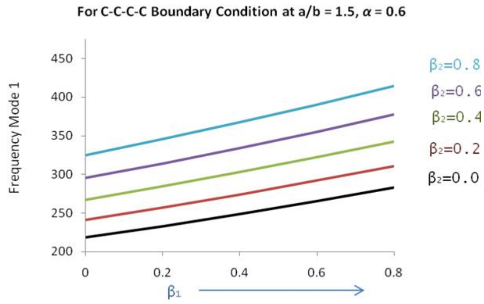

Figure 6. *Frequency (Mode 1) Vs β_1 & β_2 at α=0.6, a/b=1.5 for clamped boundary*

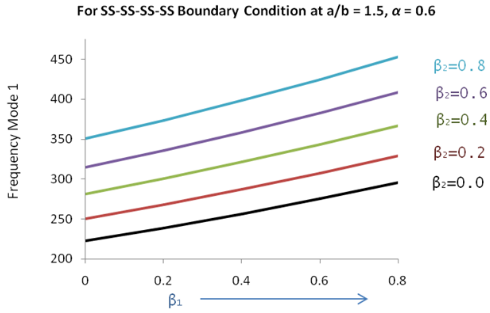

Figure 7. *Frequency (Mode 1) Vs β_1 & β_2 at α=0.6, a/b=1.5 for simply supported boundary*

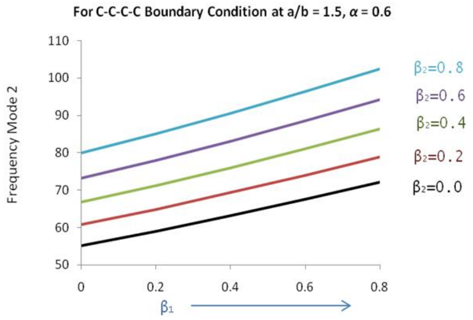

Figure 8. *Frequency (Mode 2) Vs β_1 & β_2 at α=0.6, a/b=1.5 for clamped boundary*

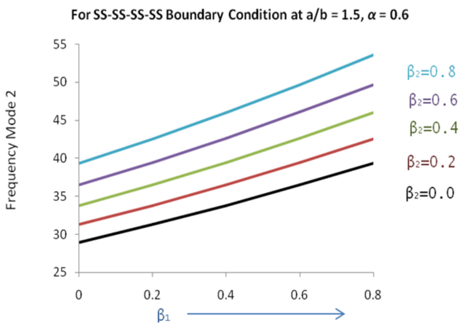

Figure 9. *Frequency (Mode 2) Vs β_1 & β_2 at α=0.6, a/b=1.5 for simply supported boundary*

For increasing values of aspect ratio, frequency for first two modes of vibration are calculated at different combinations of structural parameters for both boundary conditions and represented with the help of **Fig. 10** to **13**. At each fixed value of aspect ratio, both the modes of frequency increase continuously with increasing values of structural parameters. As aspect ratio increases, a rapid increment in both the modes of frequency is noticed. Again mode 1 of frequency for clamped boundary condition is lesser than mode 1 for simply supported boundary condition but mode 2 of frequency for clamped boundary condition is greater than mode 2 for simply supported boundary condition.

The author also observed that the frequency for both modes is lowest when the structural parameters $(\alpha, \beta_1 \ \& \ \beta_2)$ are zero, compared to when they are non-zero. The rate of change in frequency with increasing aspect ratio is significantly higher compared to the rate of change with increasing taper constants.

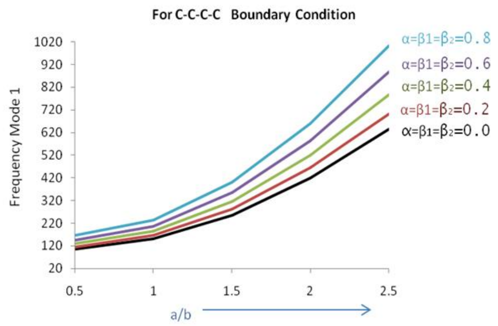

Figure 10. *Frequency (Mode 1) Vs Aspect Ratio for increasing values $\alpha=\beta_1=\beta_2$ for clamped boundary*

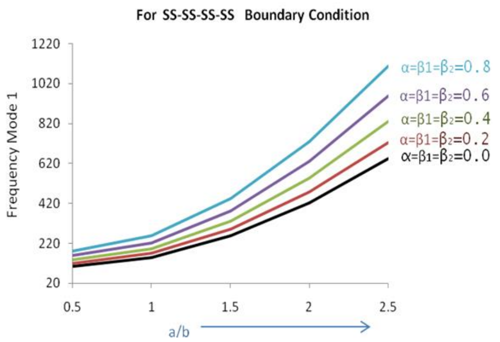

Figure 11. *Frequency (Mode 1) Vs Aspect Ratio for increasing values $\alpha=\beta_1=\beta_2$ for simply supported boundary*

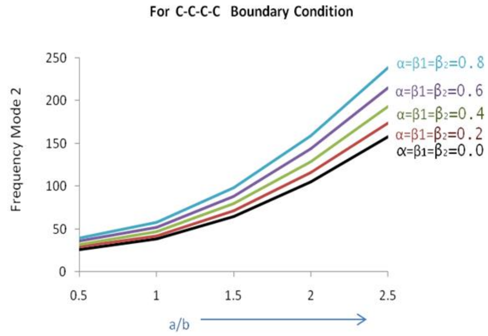

Figure 12. *Frequency (Mode 2) Vs Aspect Ratio for increasing values $\alpha = \beta_1 = \beta_2$ for clamped boundary*

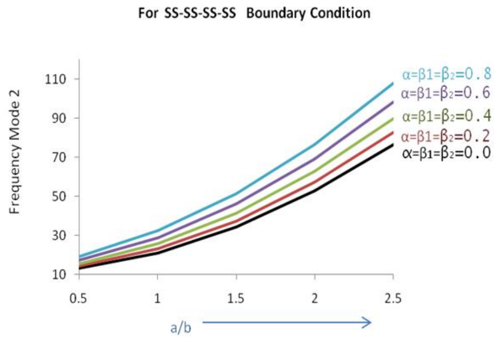

Figure 13. *Frequency (Mode 2) Vs Aspect Ratio for increasing values $\alpha = \beta_1 = \beta_2$ for simply supported boundary*

IV. CONCLUSIONS

On the basis of results & discussion, author finally concluded the present investigation with the following remarks:

1. Tapering in the thickness of the rectangular plate directly affects the frequency for first two modes of vibration. Frequency is minimum for homogeneous (non-tapered) rectangular plate for both the modes of vibration.
2. Higher tapering shows higher values of frequency. With the help of the figures, intermediate values of frequency for both the modes can be obtained for any value of taper constants.
3. A sensible monitoring of thermal gradient may help to get the desired values of frequency at corresponding values of structural parameters. Present study proves that frequency is maximum in the absence of thermal effect i.e. α =0.0.
4. Vibration of rectangular plate can be regulated by changing its dimensions i.e. aspect ratio. It shows that significance of aspect ratio may not be neglected.

5. Conditions on boundary of rectangular plate also impact on vibrational characteristics of the plate. The present mathematical model proves that first mode of frequency for simply supported boundary condition is greater than the first mode of frequency for clamped boundary condition; on the other hand, the second mode of frequency for simply supported boundary is lesser than the second mode of frequency for clamped boundary at fixed aspect ratio.

Author contributions

N. Kaur: Evaluation, Writing, and Editing.

A. Khanna: Theoretical Analysis and Supervision.

Disclosure statement

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

ORCID

N. Kaur http://orcid.org/0000-0002-6175-5680

A. Khanna http://orcid.org/0000-0003-2265-9271

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