Study of the TP transformation via the inverted pendulum example

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Abstract: This study decribes the Tensor Product (TP) based model construction through the nonlinear dynamic system of the inverted pendulum. It presents the steps of TP modeling, various weighting functions and the Linear Matrix Inequality (LMI) based approach. LMI control has been used to stabilize the nonlinear system. This study shows the quasi-Linear Parameter-Varying (qLPV) state-space modeling and the Higher-Order Singular Value Decomposition (HOSVD) based TP model transformation. Some research in this issue already exists, but only the furuta, rotary, single and parallel-type double pendulum have been examined. In this paper the TP model transformation of the inverted pendulum is analyzed in terms of stability.

Keywords: *tensor product based control; nonlinear dynamic systems; linear matrix inequality; inverted pendulum*

1. Introduction

Modeling and control of complicated nonlinear systems with multiple objectives are actually a challenge in control engineering. Many researchers investigate the H_2, H_∞ [1], LQ-optimization, pole-placement [2] methods in control theory, which can be solved by LMI based approaches. Application of LMIs can reduce a wide species of problems appearing in systems and control engineering [3].

This paper introduces a qLPV model [4] of the inverted pendulum. The LPV systems [5] approach the nonlinear system in linear way, however the properties of

the qLPV system are equivalent the nonlinear properties. It enables the representation of the nonlinear system without a complicated linearization method. LPV systems appear in the structure of Linear-Time Invariant (LTI) state-space representations. In this paper the LTI vertex system of the problem, the LMI design technique through the Parallel Distributed Compensation (PDC) and the HOSVD based TP transformation have been used. Close to Normalised (CNO) type weighting functions for an example have been applied MATLAB TP Toolbox [6]. TP transformation of pendulum has been investigated in some research for example rotary [7], furuta [8], single [9] [10] and parallel-type double pendulum [11]. This example is a good task to examine the TP model transformation and the stability of the nonlinear system.

The paper is organised as follows: in Section 2, the TP model transformation has introduced through the HOSVD based approach, the weighting functions and the qLPV model. Section 3 presents the LMI based controller design to obtain stable closed loop system based on the Lyapunov stability conditions. Due to nonlinearity, stable controller has been designed via LMI stability conditions. In Section 4, TP model transformation of the inverted pendulum has been presented, based on the Section 2-3.

2. TP model transformation

Control design of the TP model transformation was proposed by Baranyi et al. [12] [13]. The TP model structure [14] represents a multivariable tensor function. In fact, the TP model representation is a finite element convex polytopic representation [15] [16] [17]. In this paper the nonlinear system has approximated by the TP model transformation. The mathematical structure of this method is based on the HOSVD theory.

Singular value decomposition (SVD) [18] is a two dimensional matrix method used in the field of linear algebra. It is widespread in mathematics but this method is not suitable for analyzing multidimensional data. For multidimensional problems HOSVD methods [19] can be used. Multidimensional arrays or tensors represent data structures with more than two dimensions.

The HOSVD based computation extends its application to the qLPV model's continuos multivariable function. This produces numerically the TP model. TP transformation of HOSVD based approach generates the weighting funcions and the LTI vertex systems. The steps of transformating the TP model are the following [13]:

- The given qLPV model must be discretized, which means the system matrix $\mathbf{S}(\mathbf{p})$ is converted numerically into a tensor representation, where $\mathbf{p} = \mathbf{p}(t) \in \Omega$ is a time-varying parameter vector within the closed hypercube $\Omega = [a_1, b_1] \times [a_2, b_2] \times ... \times [a_n, b_n] \subset \mathbb{R}^n$. At first the problem space Ω is defined, which is a closed hypercube and the TP model is interpretable only in this space. M denotes the discretization grid defined in Ω.

- The TP model sctructure is extracted via the HOSVD structure to obtain the tensor product form.

- Weighting functions are defined. It is a computational method on HOSVD that can be converted numerically the qLPV model into that frame. The qLPV state-space representation is able to describe nonlinear systems because of combinating the LTI vertices and the nonlinear weighting functions $w_i = w_i(\mathbf{p}(t))$ i.e. weighting functions depend on the parameters. Types of the weighting functions are [13] [14]:

  - (SN) Sum Normalised: the weighting function is SN if the sum of weighting functions for all $\mathbf{p}(t) \in \Omega$ is equal to 1;

  - (NN) Non-Negative: values of the weighting functions are non- negative;

  - (NO) Normalised: if the weighting function is SN and NN type and its largest value is 1;

  - (CNO) Close to Normalised: if it is SN and NN type and its largest value is 1 or close to 1;

  - (INO) Inverse Normalised: the weighting function is INO if its smallest value is 0;

  - (RNO) Relaxed Normalised: the weighting function is RNO if its largest value is SN and NN type and that value is only between 0 and 1;

  - (IRNO) Inverse Relaxed Normalised: the weighting function is IRNO if its smallest value is 0 and its largest value is SN and NN type and that value is only between 0 and 1.

The conditions of convexity is not satisfied via HOSVD method [20] [21]. Thus, the convex hull created by the TP vertex system must be manipulated. The CNO type weighting function is used in this work to determine TP transformation of the inverted pendulum, see in Section 4.

The continuous systems [22] are written by this form:

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{p}(t))\mathbf{x}(t) + \mathbf{B}(\mathbf{p}(t))\mathbf{u}(t), \tag{1}$$

$$\mathbf{y}(t) = \mathbf{C}(\mathbf{p}(t))\mathbf{x}(t) + \mathbf{D}(\mathbf{p}(t))\mathbf{u}(t), \tag{2}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector, $\mathbf{u}(t) \in \mathbb{R}^m$ is the input vector, $\mathbf{y}(t) \in \mathbb{R}^q$ is the output vector [14]. Equations (1) and (2) can be defined by combining LTI systems. Thus, the representation can be written for instance:

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{r} w_i(\mathbf{p}(t))(\mathbf{A}_i\mathbf{x} + \mathbf{B}_i\mathbf{u}), \tag{3}$$

$$\mathbf{y}(t) = \sum_{i=1}^{r} w_i(\mathbf{p}(t))(\mathbf{C}_i\mathbf{x} + \mathbf{D}_i\mathbf{u}), \tag{4}$$

where $\mathbf{A_i}(t) \in \mathbb{R}^{n \times n}$, $\mathbf{B_i}(t) \in \mathbb{R}^{n \times m}$, $\mathbf{C_i}(t) \in \mathbb{R}^{q \times n}$, $\mathbf{D_i}(t) \in \mathbb{R}^{q \times m}$ and r is the vertex number. Equation (3) is the TP transformation. Weighting functions w_i have two important properties that satisfy convexity [13]:

$$w_i(\mathbf{p}(t)) \in [0,1], \tag{5}$$

$$\sum_{i=1}^{r} w_i(\mathbf{p}(t)) = 1. \tag{6}$$

If parameter $\mathbf{p}(t)$ does not contain elements of $\mathbf{x}(t)$, it is an LPV system. If parameter $\mathbf{p}(t)$ contains elements of $\mathbf{x}(t)$, it is a qLPV model because of the nonlinearity. The qLPV model representation of state-space configuration (1) is:

$$\mathbf{S}(\mathbf{x}(t)) = \begin{bmatrix} \mathbf{A}(\mathbf{x}(t)) & \mathbf{B}(\mathbf{x}(t)) \\ \mathbf{C}(\mathbf{x}(t)) & \mathbf{D}(\mathbf{x}(t)) \end{bmatrix}, \tag{7}$$

i.e.

$$\begin{pmatrix} \dot{\mathbf{x}} \\ \mathbf{y} \end{pmatrix} = \mathbf{S}(\mathbf{p}) \begin{pmatrix} \mathbf{x} \\ \mathbf{u} \end{pmatrix}. \tag{8}$$

3. LMI based optimization

In control science, application of LMI, especially in the field of dynamic system is related to Lyapunov [23]. This section presents the LMI based [2] control design approach. Many problems arise in control system design that are reduced by researchers to convex or quasi-convex optimization problem involving LMIs. LMIs are applied to solve many automation problems, optimization problems and system identification that are generally difficult to solve but it can be possible to solve by convex optimizing.

LMIs can solve in polynomial time. Many variety of systems and control problems can revise as LMI problems. These problems may be state-feedback synthesis, robustness design, H_2 and H_∞ control [2]. It is reducible to convex problems. The variables are often matrices, therefore application of Lyapunov-inequality is the basic structure of the LMI method [31]:

$$\mathbf{A}^T \mathbf{P} \mathbf{A} - \mathbf{P} \prec 0, \tag{9}$$

where $\mathbf{P} = \mathbf{P}^T$ is the variable. Finding $\mathbf{K}$ is the problem of control design. The closed-loop system is quadratically stable:

$$[\mathbf{A} - \mathbf{B}\mathbf{K}]^T \mathbf{P} [\mathbf{A} - \mathbf{B}\mathbf{K}] - \mathbf{P} \prec 0, \tag{10}$$

The above inequality is multiplied on the left and right via $\mathbf{P}^{-1}$, and $\mathbf{X} = \mathbf{P}^{-1}$ $\mathbf{M} = \mathbf{K}\mathbf{X}$:

$$\mathbf{P}^{-1}[\mathbf{A} - \mathbf{B}\mathbf{K}]^T \mathbf{P} [\mathbf{A} - \mathbf{B}\mathbf{K}] \mathbf{P}^{-1} - \mathbf{P}^{-1} \mathbf{P} \mathbf{P}^{-1} \prec 0, \tag{11}$$

i.e.

$$\mathbf{X}[\mathbf{A} - \mathbf{BK}]^T \mathbf{X}^{-1}[\mathbf{A} - \mathbf{BK}]\mathbf{X} - \mathbf{X} \prec 0, \tag{12}$$

finally

$$\mathbf{X} - [\mathbf{AX} - \mathbf{BM}]^T \mathbf{X}^{-1}[\mathbf{AX} - \mathbf{BM}] \succ 0. \tag{13}$$

It can be transformed into the LMI form:

$$\begin{bmatrix} \mathbf{X} & (\mathbf{AX} - \mathbf{BM})^T \\ (\mathbf{AX} - \mathbf{BM}) & \mathbf{X} \end{bmatrix} \succ 0. \tag{14}$$

This form is defined by Schur complements. Defining the Schur complements [2] of a partitioned $\mathbf{A}$ matrix can be written as:

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{15}$$

For example if A_{11} is nonsingular, $A_{22} - A_{21}A_{11}^{-1}A_{12}$ is the Schur complement of A_{11}. It can be denoted by $S_{ch}(A_{11})$. Thus, matrix $\mathbf{A}$ can be written as:

$$\mathbf{A} = \begin{bmatrix} A_{11} & 0 \\ 0 & S_{ch}(A_{11}) \end{bmatrix}. \tag{16}$$

3.1. Presenting LMI based approach

PDC is a model based design procedure that was suggested by Tanaka and Wang [24]. They designed for Takagi-Sugeno (TS) fuzzy decision [31] models so their procedure was to develop a fuzzy controller from a TS fuzzy model. Combination of LMI optimization and PDC framework can be accomplished with convex optimization. The PDC design structures are defined one LTI feedback gain $\mathbf{k}$ to all LTI vertex systems.

Therefore, the TP controller is:

$$\mathbf{u}(t) = -\sum_{i=1}^{r} w_i(\mathbf{p}(t))\mathbf{k}_i\mathbf{x}(t) = -\left(\sum_{i=1}^{r} w_i(\mathbf{p}(t))\mathbf{k}_i\right)\mathbf{x}(t) = -\mathbf{k}^T\mathbf{x}(t). \quad (17)$$

Substituting (17) into (3) the following representation is obtained:

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{r} w_i[\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i(-\sum_{j=1}^{r} w_j\mathbf{k}_j\mathbf{x}(t))], \quad (18)$$

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} w_iw_j[\mathbf{A}_i - \mathbf{B}_i\mathbf{k}_j]\mathbf{x}(t), \quad (19)$$

where $i,j = 1...r$, r is the total number of LTI systems. It can be rewritten into this form:

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{r} w_iw_i[\mathbf{A}_i - \mathbf{B}_i\mathbf{k}_i]\mathbf{x}(t) + 2\sum_{i=1}^{r}\sum_{i<j}^{r} w_iw_j\mathbf{G}_{ij}\mathbf{x}(t), \quad (20)$$

where $\mathbf{G}$ is:

$$\mathbf{G}_{ij} = \frac{[\mathbf{A}_i - \mathbf{B}_i\mathbf{k}_j] + [\mathbf{A}_j - \mathbf{B}_j\mathbf{k}_i]}{2}. \quad (21)$$

3.2.   The Lyapunov theory

The LMI based approach began with Lyapunov [23] based stability conditions. In the case of nonlinear dynamic systems, stability analysis methods developed for linear systems, are not applicable. The Lyapunov methods are used to get suiffcent conditions for stability of the equilibrium point of a linear system.

Consider a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},t)$, where $\mathbf{f}$ is continuos. To study the stability of nonlinear dynamic systems, the Lyapunov theorem can be applied. If there exists a function $V(\mathbf{x},t)$ what can be derived around an equilibrium point $\mathbf{x} = 0$ and satisfies the following conditions:

- $V(\mathbf{0},t) = 0$ and $V(\mathbf{x},t) > 0$ if $V(\mathbf{x},t)$ is a positive defitine function,
- $\dot{V}(\mathbf{x},t) \leq 0$, i.e. $\dot{V}(\mathbf{x},t)$ is a negative semidefinite function,
- $\dot{V}(\mathbf{x},t) < 0$, i.e. $\dot{V}(\mathbf{x},t)$ is a negative definite function.

If first and second conditions are existed, then the system is stable, and if the first and third conditions are existed the system is asymptotically stable. The function V in Lyapunov theorem is the energy function assigned to the system.

Consider a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{u})$ with control inputs u, where $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{u}(t) \in \mathbb{R}^m$. Consider the candidate Lyapunov function $V(\mathbf{x})$ for every fixed $\mathbf{x} \neq 0$ there exists and acceptable value $\mathbf{u}$ for the control in such a way as to:

$$\nabla V(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x},\mathbf{u}) < 0. \tag{22}$$

Regard the following candidate Lyapunov function [31] $V(\mathbf{x}(t)) = \mathbf{x}^T(t)\mathbf{P}\mathbf{x}(t)$ where $\mathbf{P} \succ 0$, thus:

$$
\begin{aligned}
\dot{\mathbf{V}}(\mathbf{x}(t)) &= \sum_{i=1}^{r}\sum_{j=1}^{r} w_i w_j \mathbf{x}^T(t)[(\mathbf{A}_i - \mathbf{B}_i\mathbf{k}_j)^T \mathbf{P} + \mathbf{P}(\mathbf{A}_i - \mathbf{B}_i\mathbf{k}_j)]\mathbf{x}(t) \\
&= \sum_{i=1}^{r} w_i^2 \mathbf{x}^T(t)[\mathbf{G}_{ii}^T\mathbf{P} + \mathbf{P}\mathbf{G}_{ii}]\mathbf{x}(t) + \sum_{i=1}^{r}\sum_{i<j} 2w_i w_j \mathbf{x}^T(t) \\
&\times \left[\left(\frac{\mathbf{G}_{ij} + \mathbf{G}_{ji}}{2}\right)^T \mathbf{P} + \mathbf{P}\left(\frac{\mathbf{G}_{ij} + \mathbf{G}_{ji}}{2}\right)\right]\mathbf{x}(t).
\end{aligned} \tag{23}
$$

3.3. Stable controller design with LMI based stability conditions

Consider the stability of an open-loop system via Lyapunov stability concept, where the equilibrium of TP system (19) via $\mathbf{u}(t) = 0$ is globally asymptotically stable, then there is a common positive definite matrix $\mathbf{P}$ for instance

$$\mathbf{A}_i^T\mathbf{P} + \mathbf{P}\mathbf{A}_i \prec 0, \tag{24}$$

for $\forall i = 1, \cdots, r$. Examine stability of an closed-loop system, as:

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{r} \sum_{j=1}^{r} w_i w_j (\mathbf{A}_i - \mathbf{B}_i \mathbf{K}_j) \mathbf{x}(t), \tag{25}$$

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{r} w_i w_i [\mathbf{A}_i - \mathbf{B}_i \mathbf{K}_i] \mathbf{x}(t) + 2 \sum_{i=1}^{r} \sum_{i<j}^{r} w_i w_j \mathbf{G}_{ij} \mathbf{x}(t), \tag{26}$$

Let us use the following notations:

$$\mathbf{G}_{ij} = \mathbf{A}_i - \mathbf{B}_i \mathbf{K}_j, \tag{27}$$

$$\mathbf{G}_{ji} = \mathbf{A}_j - \mathbf{B}_j \mathbf{K}_i, \tag{28}$$

where $i = 1, \cdots, r$. The equilibrium point of the TP system (9) is globally asymptotically stable if there is a common positive definite matrix $\mathbf{P}$:

$$\mathbf{G}_{ii}^T \mathbf{P} + \mathbf{P} \mathbf{G}_{ii} \prec 0, \tag{29}$$

and

$$\left(\frac{\mathbf{G}_{ij} + \mathbf{G}_{ji}}{2} \right)^T \mathbf{P} + \mathbf{P} \left(\frac{\mathbf{G}_{ij} + \mathbf{G}_{ji}}{2} \right) \preceq 0. \tag{30}$$

Multiplying inequelity (29) on the left and right by $\mathbf{P}^{-1}$ and denote $\mathbf{X} = \mathbf{P}^{-1}$ and $\mathbf{M_i} = \mathbf{K_i X}$:

$$\left(\mathbf{P}^{-1} \mathbf{A}_i \mathbf{P} - \mathbf{P}^{-1} \mathbf{B}_i \mathbf{K}_i \mathbf{P} \right)^T + \left(\mathbf{P}^{-1} \mathbf{P} \mathbf{A}_i - \mathbf{P}^{-1} \mathbf{P} \mathbf{B}_i \mathbf{K}_i \right) \prec 0, \tag{31}$$

finally

$$\left(\mathbf{P}^{-1}\mathbf{A}_i\mathbf{P}\mathbf{P}^{-1} - \mathbf{P}^{-1}\mathbf{B}_i\mathbf{K}_i\mathbf{P}\mathbf{P}^{-1}\right)^T$$
$$+ \left(\mathbf{P}^{-1}\mathbf{P}\mathbf{A}_i\mathbf{P}^{-1} - \mathbf{P}^{-1}\mathbf{P}\mathbf{B}_i\mathbf{K}_i\mathbf{P}^{-1}\right) \prec 0, \tag{32}$$

Substituting $\mathbf{X} = \mathbf{P}^{-1}$ and $\mathbf{M_i} = \mathbf{K_i}\mathbf{X}$, then:

$$-\mathbf{X}\mathbf{A}_i^T - \mathbf{A}_i\mathbf{X} + \mathbf{X}\mathbf{K}_i^T\mathbf{B}_i^T + \mathbf{B}_i\mathbf{K}_i\mathbf{X} \succ 0, \tag{33}$$

the following LMI stability condition is got:

$$-\mathbf{X}\mathbf{A}_i^T - \mathbf{A}_i\mathbf{X} + \mathbf{M}_i^T\mathbf{B}_i^T + \mathbf{B}_i\mathbf{M}_i \succ 0. \tag{34}$$

Equations (27) and (28) are applied, and denote $\mathbf{X} = \mathbf{P}^{-1}$ and $\mathbf{M_i} = \mathbf{K_i}\mathbf{X}$:

$$\left(\frac{\mathbf{A}_i - \mathbf{B}_i\mathbf{K}_j + \mathbf{A}_j - \mathbf{B}_j\mathbf{K}_i}{2}\right)^T \mathbf{P} + \mathbf{P}$$
$$\left(\frac{\mathbf{A}_i - \mathbf{B}_i\mathbf{K}_j + \mathbf{A}_j - \mathbf{B}_j\mathbf{K}_i}{2}\right) \preceq 0, \tag{35}$$

then multiplying inequelity (30) on the left and right by $\mathbf{P}^{-1}$, then the following equation is got:

$$\left(\frac{\mathbf{A}_i\mathbf{P} - \mathbf{B}_i\mathbf{K}_j\mathbf{P} + \mathbf{A}_j\mathbf{P} - \mathbf{B}_j\mathbf{K}_i\mathbf{P}}{2}\right)^T$$
$$+ \left(\frac{\mathbf{P}\mathbf{A}_i - \mathbf{P}\mathbf{B}_i\mathbf{K}_j + \mathbf{P}\mathbf{A}_j - \mathbf{P}\mathbf{B}_j\mathbf{K}_i}{2}\right) \preceq 0, \tag{36}$$

$$\left(\frac{\mathbf{P}^{-1}\mathbf{A}_i\mathbf{P} - \mathbf{P}^{-1}\mathbf{B}_i\mathbf{K}_j\mathbf{P} + \mathbf{P}^{-1}\mathbf{A}_j\mathbf{P} - \mathbf{P}^{-1}\mathbf{B}_j\mathbf{K}_i\mathbf{P}}{2}\right)^T$$

$$+ \left(\frac{\mathbf{P}^{-1}\mathbf{P}\mathbf{A}_i - \mathbf{P}^{-1}\mathbf{P}\mathbf{B}_i\mathbf{K}_j + \mathbf{P}^{-1}\mathbf{P}\mathbf{A}_j - \mathbf{P}^{-1}\mathbf{P}\mathbf{B}_j\mathbf{K}_i}{2}\right) \preceq 0, \tag{37}$$

$$\left(\frac{\mathbf{P}^{-1}\mathbf{A}_i\mathbf{P}\mathbf{P}^{-1} - \mathbf{P}^{-1}\mathbf{B}_i\mathbf{K}_j\mathbf{P}\mathbf{P}^{-1} + \mathbf{P}^{-1}\mathbf{A}_j\mathbf{P}\mathbf{P}^{-1} - \mathbf{P}^{-1}\mathbf{B}_j\mathbf{K}_i\mathbf{P}\mathbf{P}^{-1}}{2}\right)^T$$

$$+ \left(\frac{\mathbf{P}^{-1}\mathbf{P}\mathbf{A}_i\mathbf{P}^{-1} - \mathbf{P}^{-1}\mathbf{P}\mathbf{B}_i\mathbf{K}_j\mathbf{P}^{-1} + \mathbf{P}^{-1}\mathbf{P}\mathbf{A}_j\mathbf{P}^{-1} - \mathbf{P}^{-1}\mathbf{P}\mathbf{B}_j\mathbf{K}_i\mathbf{P}^{-1}}{2}\right) \preceq 0, \tag{38}$$

Substituting $\mathbf{X} = \mathbf{P}^{-1}$ and $\mathbf{M_i} = \mathbf{K_i}\mathbf{X}$:

$$\left(\frac{\mathbf{X}\mathbf{A}_i - \mathbf{X}\mathbf{B}_i\mathbf{K}_j + \mathbf{X}\mathbf{A}_j - \mathbf{X}\mathbf{B}_j\mathbf{K}_i}{2}\right)^T$$

$$+ \left(\frac{\mathbf{A}_i\mathbf{X} - \mathbf{B}_i\mathbf{K}_j\mathbf{X} + \mathbf{A}_j\mathbf{X} - \mathbf{B}_j\mathbf{K}_i\mathbf{X}}{2}\right) \preceq 0, \tag{39}$$

$$-\mathbf{X}\mathbf{A}_i^T - \mathbf{A}_i\mathbf{X} - \mathbf{X}\mathbf{A}_j^T - \mathbf{A}_j\mathbf{X} + \mathbf{X}\mathbf{K}_j^T\mathbf{B}_i^T + \mathbf{B}_i\mathbf{K}_j\mathbf{X} + \mathbf{X}\mathbf{K}_i^T\mathbf{B}_j^T + \mathbf{B}_j\mathbf{K}_i\mathbf{X} \succ 0, \tag{40}$$

after this, another LMI stability condition is got:

$$-\mathbf{X}\mathbf{A}_i^T - \mathbf{A}_i\mathbf{X} - \mathbf{X}\mathbf{A}_j^T - \mathbf{A}_j\mathbf{X} + \mathbf{M}_j^T\mathbf{B}_i^T + \mathbf{B}_i\mathbf{M}_j + \mathbf{M}_i^T\mathbf{B}_j^T + \mathbf{B}_j\mathbf{M}_i \succeq 0. \tag{41}$$

Equations (34) and (41) are the LMI stability conditions and these are applied for the inverted pendulum example by equations (66) and (67).

4. TP transformation of the inverted pendulum

The center of mass of the inverted pendulum [25] is above its pivot point. Control designing of this nonlinear system has been the subject of many studies and researchers in the recent period [26] [27] [28].
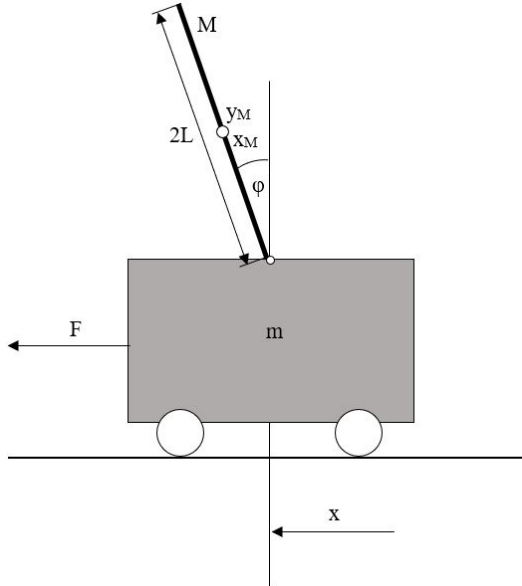

Figure 1. Inverted Pendulum

4.1. Modeling of inverted pendulum

The illustration of the inverted pendulum installed on a cart is given in Fig 1. The lenght of the rod on the car is $2L = 0.35$, the mass of the rod is $M = 1.2$, the mass of the cart $m = 0.2$ in a coherent unit system SI and the generalized coordinates are x and φ. The system equations of the nonlinear system are written by Euler-Lagrange

equation [29]:

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_i} - \frac{\partial K}{\partial q_i} + \frac{\partial P}{\partial q_i} = \tau_i, \tag{42}$$

where K is the kinetic energy, P is the potential energy, q_i is the generalized coordinate and τ_i is the generalized force. The moment of inertia of the rod belonging to the center of mass is $\Theta = \frac{ML^2}{3}$. The center of gravity of the rod mass is given by the coordinates $x_M = x + LS_\varphi$ and $y_M = LC_\varphi$ and the velocity is v_M. Let's denote $S_\varphi = \sin(\varphi)$ and $C_\varphi = \cos(\varphi)$ [29] [26] [32]. After derivating x_M and y_M, the following results can be got:

$$\dot{x}_M = \dot{x} + \dot{\varphi}LC_\varphi, \tag{43}$$

$$\dot{y}_M = -\dot{\varphi}LS_\varphi, \tag{44}$$

$$v_M^2 = \dot{x}_M^2 + \dot{y}_M^2 = \dot{x}^2 + 2LC_\varphi\dot{x}\dot{\varphi} + L^2\dot{\varphi}^2. \tag{45}$$

Determining kinetic energy and potential energy as follows:

$$\begin{aligned} K &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}Mv_M^2 + \frac{1}{2}\Theta\dot{\varphi}^2 \\ &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}M(\dot{x}^2 + 2LC_\varphi\dot{x}\dot{\varphi} + L^2\dot{\varphi}^2) + \frac{1}{2}\Theta\dot{\varphi}^2, \end{aligned} \tag{46}$$

$$P = MgLC_\varphi. \tag{47}$$

After obtaining the partial derivatives of K and P, and applying mathematical manipulations, the following equations are given:

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{x}} = (m + M)\ddot{x} + MLC_\varphi\ddot{\varphi} - MLS_\varphi\dot{\varphi}^2, \tag{48}$$

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{\varphi}} = MLC_\varphi \ddot{x} - MLS_\varphi \dot{x}\dot{\varphi} + (\Theta + ML^2)\ddot{\varphi}. \tag{49}$$

Using Euler-Lagrange equation (42), the following equations are got:

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{x}} - \frac{\partial K}{\partial x} + \frac{\partial P}{\partial x} = F, \tag{50}$$

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{\varphi}} - \frac{\partial K}{\partial \varphi} + \frac{\partial P}{\partial \varphi} = 0, \tag{51}$$

i.e.

$$(M + m)\ddot{x} + MLC_\varphi \ddot{\varphi} - MLS_\varphi \dot{\varphi}^2 = F, \tag{52}$$

$$MLC_\varphi \ddot{x} + (\Theta + ML^2)\ddot{\varphi} - MgLS_\varphi = 0. \tag{53}$$

After some mathematical manipulations the following equations are given:

$$\ddot{x} = \frac{F - MLC_\varphi \ddot{\varphi} + MLS_\varphi \dot{\varphi}^2}{m + M}, \tag{54}$$

$$\ddot{\varphi} = \frac{gS_\varphi - \frac{C_\varphi}{m+M}(F + MLS_\varphi \dot{\varphi}^2)}{L\left(\frac{4}{3} - \frac{MC_\varphi^2}{m+M}\right)}. \tag{55}$$

Then the following state-space representation is defined:

$$\dot{x}_1 = x_2, \tag{56}$$

$$\dot{x}_2 = \frac{\frac{4}{3}MLsin(x_3)x_4^2 - Mgcos(x_3)sin(x_3) + \frac{4}{3}F}{\frac{4}{3}(m + M) - Mcos^2(x_3)}, \tag{57}$$

$$\dot{x}_3 = x_4, \tag{58}$$

$$\dot{x}_4 = \frac{MLsin(x_3)cos(x_3)x_4^2 + cos(x_3)F - (m + M)gsin(x_3)}{MLsin^2(x_3) - \frac{4}{3}(m + M)L}, \quad (59)$$

where the state-space variables are $x_1 = x$, $x_2 = \dot{x}$, $x_3 = \varphi$, $x_4 = \dot{\varphi}$.

4.2. TP transformation

It is an important purpose of transforming TP to convert the given state-space model (1) into a convex TP model. In this model the LTI systems create a tight convex hull [30]. Consider the following state-space structure:

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{x})\mathbf{x} + \mathbf{B}(\mathbf{x})u, \quad (60)$$

Elements of matrix $\mathbf{A}(\boldsymbol{x})$ (61) and vector $\mathbf{B}(\mathbf{x})$ (62) depend on only the state variables x_3 and x_4, i.e. nonlinearity is caused by x_3 and x_4. For TP model transfromation the transformation space $\Omega = [(-45/180)\pi, (45/180)\pi] \times [(-45/180)\pi, (45/180)\pi]$ is defined and discretized by $M_1 \times M_2 = 136 \times 136$ grid points. The components of matrix $\mathbf{A}(\boldsymbol{x})$ and vector $\mathbf{B}(\mathbf{x})$ have been defined numerically and they are stored in a four dimensional tensor $S \in R^{136 \times 136 \times 4 \times 5}$,

$$\mathbf{A}(\boldsymbol{x}) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-Mgcos(x_3)}{\frac{4}{3}(m+M)-Mcos^2(x_3)} & \frac{sin(x_3)}{x_3} & \frac{\frac{4}{3}MLsin(x_3)}{\frac{4}{3}(m+M)-Mcos^2(x_3)}x_4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{(m+M)g}{MLsin^2(x_3)-\frac{4}{3}(m+M)L} & \frac{sin(x_3)}{x_3} & \frac{MLsin(x_3)cos(x_3)}{MLsin^2(x_3)-\frac{4}{3}(m+M)L}x_4 \end{bmatrix}, \quad (61)$$

$$\mathbf{B}(\mathbf{x}) = \begin{bmatrix} 0 \\ \frac{\frac{4}{3}}{\frac{4}{3}(m+M)-Mcos^2(x_3)} \\ 0 \\ \frac{cos(x_3)}{MLsin^2(x_3)-\frac{4}{3}(m+M)L} \end{bmatrix}. \quad (62)$$

Applying HOSVD, the singular values are $\sigma_1 = 1909$, $\sigma_2 = 201$, $\sigma_3 = 23$, $\sigma_4 = 2$ for the variable x_3 and $\sigma_1 = 1920$, $\sigma_2 = 23$ for the variable x_4. After using HOSVD

on $\mathbf{S}(\mathbf{x}(t))$ and CNO type weighting function, the following TP model transformation describes inverted pendulum system:

$$\dot{\mathbf{x}} \cong \sum_{i_1=1}^{r_1} \sum_{i_2=1}^{r_2} w_{i_1}(x_3) w_{i_2}(x_4) (\mathbf{A}_{i_1,i_2} \mathbf{x} + \mathbf{B}_{i_1,i_2} u), \tag{63}$$

where $w_{i_1}(x_3)$ and $w_{i_2}(x_4)$ are the CNO type weighting functions, $\mathbf{A}_{i_1,i_2}$ and $\mathbf{B}_{i_1,i_2}$ are state independent system matrix and vector. This nonlinear system is possible to be approximated by the combinations of $r_1 r_2$ LTI systems. It can be written as follows:

$$\dot{\mathbf{x}} \cong \sum_{i_1=1}^{5} \sum_{i_2=1}^{2} w_{i_1}(x_3) w_{i_2}(x_4) (\mathbf{A}_{i_1,i_2} \mathbf{x} + \mathbf{B}_{i_1,i_2} u), \tag{64}$$

$$\dot{\mathbf{x}} \cong \sum_{i=1}^{r} w_i(x_3, x_4) (\mathbf{A}_i \mathbf{x} + \mathbf{B}_i u). \tag{65}$$

where $r = r_1 r_2$ is the number of LTI systems via $5 \times 2 = 10$ LTI vertex models. Weighting functions $w_{i_1}(x_3)$ and $w_{i_2}(x_4)$ are shown in Fig. 2-3. Thus, the system is decomposed in angular velocity $\dot{\varphi}$ into 2 weighting functions, and in angular position φ into 5 weighting functions.
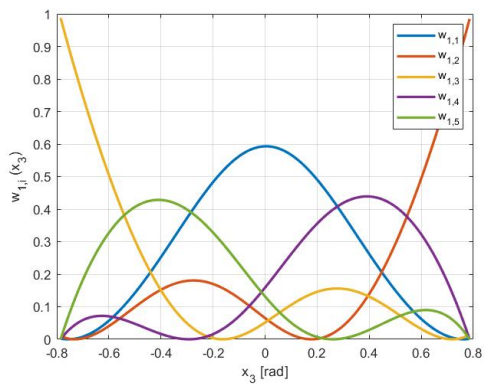
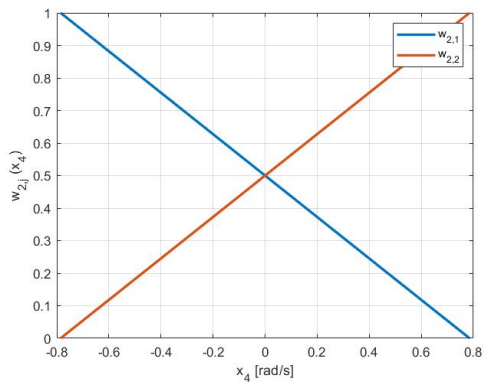*Figure 2. CNO type weighting functions for x_3*

Figure 3. CNO type weighting functions for x_4

The controller vector $\mathbf{k}_i$ can be numerically defined by LMI. Using stability conditions (31) and (41), determining asymptotic stability of the system. Thus, the stabilization controller can be found:

$$-\mathbf{X}\mathbf{A}_i^T - \mathbf{A}_i\mathbf{X} + \mathbf{M}_i^T\mathbf{B}_i^T + \mathbf{B}_i\mathbf{M}_i \succ 0, \tag{66}$$

$$-\mathbf{X}\mathbf{A}_i^T - \mathbf{A}_i\mathbf{X} - \mathbf{X}\mathbf{A}_j^T - \mathbf{A}_j\mathbf{X} + \mathbf{M}_j^T\mathbf{B}_i^T + \mathbf{B}_i\mathbf{M}_j + \mathbf{M}_i^T\mathbf{B}_j^T + \mathbf{B}_j\mathbf{M}_i \succeq 0, \tag{67}$$

$$\mathbf{k}_i = \mathbf{M}_i\mathbf{X}^{-1}, \tag{68}$$

where $i = 1, \cdots, r$ and $j = i + 1, \cdots, r$, and r is the number of LTI vertex systems.

The control signal is shown in Fig. 6. and state variables x_1 and x_3 are shown in Fig. 4-5. The initial condition is $\mathbf{x}(0) = [0.3,0,0,0]$.
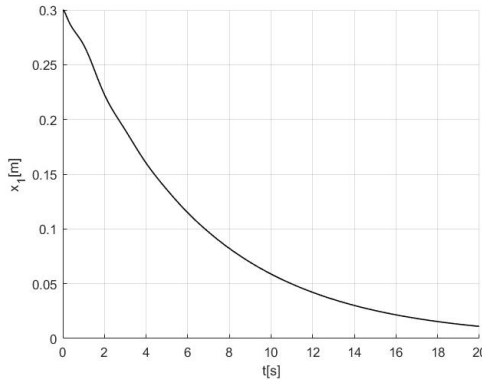

Figure 4. State variable x_1

It can be seen in Fig. 4-6. that functions x_3 and u return to a steady state after a few oscillations. The system get into the stable position around 10 seconds. This will result that LMI approach should improve to stabilize better this nonlinear system.
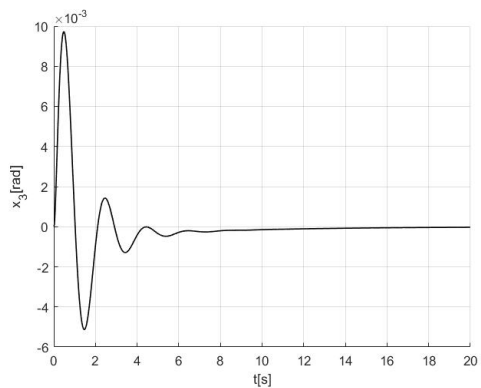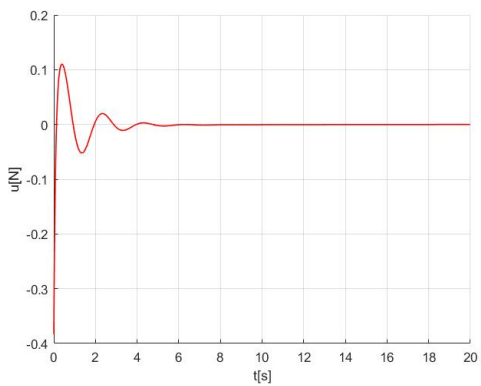
Figure 5. State variable x_3

Figure 6. Control signal u

5. Conclusion

This paper describes a study of the HOSVD based TP transformation with CNO type weighting functions using LMI solver and a solution to a problem by applying these methods through an inverted pendulum example. In this example there are 10 LTI vertex systems of the qLPV model. This paper shows an example to analyse global asymptotic stability. Therefore in the future the decay rate for the inverted pendulum and H_2, H_∞ controllers will be analysed.

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