Acta
Technica
Jaurinensis

Vol.7., No.3., pp. 319–327, 2014
DOI: 10.14513/actatechjaur.v7.n3.312
Available online at acta.sze.hu

A Fourth Order Finite Difference Method for Numerical Solution of the Goursat Problem

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Abstract: In this article, we consider the finite difference method for numerically solving the Goursat Problem, using uniform Cartesian grids on the square region. Numerical examples are considered to ensure accuracy of the developed method on both linear and nonlinear Goursat Problems of second order partial differential equations. The results obtained for these numerical examples validate the efficiency and expected fourth order accuracy of the method.

Keywords: finite difference method, fourth order method, Goursat problem, maximum absolute error, numerical method, stability

1. Introduction

The mathematical formulation of physical phenomena in natural sciences and engineering often leads to an initial value problem of partial differential equations. This type of problem can be formulated either in terms of first order PDEs or higher order PDEs. The Goursat problem arises in the study of wave phenomena. The solutions are frequently required in many applications such as accoustic scattering[1], Sine Gordon [2], electromagnetic theory [3] and wave equation [4]. Finite difference methods are commonly used to solve the Goursat problem. In addition to finite difference methods, other methods may be applied to numerically solve the Goursat problem; for example, heronian mean averaging method [5], cubature method [6], nonlinear trapezoidal method [7], adomian decomposition method [8], variational iteration method and references therein.

Developing an efficient and accurate numerical method for solutions for the Goursat problem is an active research topic. A new fourth order compact finite difference method for solving the Goursat problem was reported in [9], following the ideas therein.

In this article we consider a novel exponential finite difference approach which precisely satisfies the initial conditions. The present work is organised as follows. In section 2, we consider the development and derivation of novel exponential finite difference approximation for the Goursat problem. A novel finite difference method is presented so that the resulting difference equation need satisfies the initial conditions exactly. A local truncation error,convergence and stability of the present method discussed in section 3 and finally, the application of the developed method presented together with illustrative numerical results have been produced to show the efficiency of the method in section 4. A discussion and conclusion on the performance of the method are presented in section 5.

2. Derivation of Method

The subject of the present paper is to develop a finite difference method for the numerical solution of linear and nonlinear Goursat problem which arise in physical phenomena and applied sciences. For this purpose, we consider the Goursat problem [8],

$$\frac{\partial^2 u}{\partial x \partial y} = f(x,y,u), \qquad 0 \leq x,y \leq 1 \tag{1}$$

subject to boundary conditions

$$\begin{aligned} &u(x,0) = g(x), \quad and \quad u(0,y) = h(y) \\ &g(0) = \ h(0) \end{aligned} \tag{2}$$

This problem with a different source function $f(x,y,u,u_x,u_y)$ has been examined by several numerical methods such as cubature method [6], adomian decomposition method [8], nonlinear trapezoidal [7], Runge-Kutta method [10], finite difference method [9] and references therein.

The existence and uniqueness of the solution of the problem (1) is assumed. The specific assumption on $f(x,y,u)$ to ensure existence and uniqueness will not be considered.

We superimpose on the region of interest a mesh by lines $x_m = mh$, $y_m = mh$, $m = 0,1,2,.....N$, with mesh size $h = 1/N$ in x and y directions respectively. For convenience of notation the following symbolism is used. We denote the nodal point (x_i,y_j) as (i,j) and value of the source function f evaluated at the mesh point (x_i,y_j,u_{ij}) by f_{ij} and similarly we can define other notations in this article. Suppose we have to determine a number $u_{i+1,j+1}$, which is a numerical approximation of the theoretical value of $u(x_i + h,y_j + h)$, a solution of the problem (1) at the mesh point $(x_i + h,y_j + h)$.

To derive the method, we consider 9-points (i,j), $(i \pm 1,j)$, $(i,j \pm 1)$ and $(i \pm 1,j \pm 1)$. Following the idea in [11], we propose an approximation to theoretical solution $u(x_i +$

$h,y_j+h)$ of the problem (1) as an exponential difference method

$$u(x_i+h,y_j+h)-u(x_i+h,y_j-h)-u(x_i-h,y_j+h)+u(x_i-h,y_j-h) = b_0h^2f(x_i,y_j,u(x_i,y_j))e^{\phi(x_i+h,y_j+h)} \quad (3)$$

where b_0 is the unknown constant, $\phi(x_i+h,y_j+h)$ an unknown sufficiently differentiable function.

Let us define a function $F(x_i,y_j,u(x_i,y_j),h)$ as

$$F(x_i,y_j,u(x_i,y_j),h) \equiv u(x_i+h,y_j+h)-u(x_i+h,y_j-h)-u(x_i-h,y_j+h)+u(x_i-h,y_j-h) - b_0h^2f(x_i,y_j,u(x_i,y_j))e^{\phi(x_i+h,y_j+h)} = 0 \quad (4)$$

By Taylor series expansion of $\phi(x_i+h,y_j+h)$ about mesh point (x_i,y_j), we have

$$\phi(x_i+h,y_j+h) = \phi_{ij}+h(\phi_{xij}+\phi_{yij})+\frac{h^2}{2}(\phi_{xxij}+2\phi_{xyij}+\phi_{yyij})+O(h^3) \quad (5)$$

where $\phi(x_i,y_j)=\phi_{ij}$, $(\frac{\partial\phi}{\partial x})_{ij}=\phi_{xij}$,......etc.. The expression (5) will provide an $O(h^3)$ approximation for the function $e^{\phi(x_i+h,y_j+h)}$.
So, we have

$$e^{\phi(x_i+h,y_j+h)} = 1+\phi_{ij}+h(\phi_{xij}+\phi_{yij})+\frac{h^2}{2}(\phi_{xxij}+2\phi_{xyij}+\phi_{yyij}) + \frac{1}{2}\{\phi_{ij}^2+h^2(\phi_{xij}+\phi_{yij})^2+2h\phi_{ij}(\phi_{xij}+\phi_{yij}) + h^2\phi_{ij}(\phi_{xxij}+2\phi_{xyij}+\phi_{yyij})\}+O(h^3) \quad (6)$$

In order to determine a constant and $\phi(x_i+h,y_j+h)$, by the Taylor series expansion of the function $u(x,y)$ about the mesh point(x_i,y_j) and using notations defined above, from(4) we will write the expansion as

$$F_{ij}(x,y,u,h) \equiv h^2(4(u_{xy})_{ij}-b_0(1+\phi_{ij}+\frac{1}{2}\phi_{ij}^2)f_{ij})-b_0h^3(1+\phi_{ij})(\phi_{xij}+\phi_{yij})f_{ij} + h^4(\frac{2}{3}(u_{xxxy}+u_{xyyy})_{ij}-\frac{1}{2}b_0((\phi_{xij}+\phi_{yij})^2+(1+\phi_{ij})(\phi_{xxij}+2\phi_{xyij}+\phi_{yyij}))f_{ij} = 0 \quad (7)$$

where $\frac{\partial^2u}{\partial x\partial y}=u_{xy}$, $\frac{\partial^4u}{\partial x^3\partial y}=u_{xxxy}$, etc...

Comparing the coefficients of h^p, $p = 2,3,4$ in (7) and using the fact that $u_{xyij} = f_{ij}$ from (1), we will have following system of nonlinear equations

$$\begin{aligned} b_0(1+\phi_{ij}+\frac{1}{2}\phi_{ij}^2) &= 4 \\ b_0(1+\phi_{ij})(\phi_{xij}+\phi_{yij})f_{ij} &= 0 \\ \frac{1}{2}b_0((\phi_{xij}+\phi_{yij})^2+(1+\phi_{ij})(\phi_{xxij}+2\phi_{xyij}+\phi_{yyij}))f_{ij} & \\ &= \frac{2}{3}(u_{xxxyij}+u_{xyyyij}) \end{aligned} \tag{8}$$

To simplify the calculation and solving the system of equations (8), let us assume $\phi_{ij} = 0$, $\phi_{xij} = 0$ and $\phi_{yij} = 0$. So we will get

$$b_0 = 4 \tag{9}$$

$$\phi_{xxij} + 2\phi_{xyij} + \phi_{yyij} = \frac{1}{3f_{ij}}(u_{xxxy} + u_{xyyy})_{ij} \tag{10}$$

On Substitution of the values of ϕ_{ij} , ϕ_{xij} , ϕ_{yij} and $\phi_{xxij} + 2\phi_{xyij} + \phi_{yyij}$ in (5), and assuming the negligible contribution of the terms with $O(h^3)$, we have

$$\phi(x_i + h,y_j + h) = \frac{h^2}{6f_{ij}}(u_{xxxy} + u_{xyyy})_{ij} \tag{11}$$

Finally substituting the values of b_0 and $\phi(x_i + h,y_j + h)$ from (9-10) in (4), we will obtain the expression

$$\begin{aligned} u(x_i+h,y_j+h) - u(x_i+h,y_j-h) - u(x_i-h,y_j+h) + u(x_i-h,y_j-h) & \\ = 4h^2 f(x_i,y_j,u(x_i,y_j))e^{\frac{h^2(u_{xxxy}+u_{xyyy})_{ij}}{6f_{ij}}} \end{aligned} \tag{12}$$

This is an explicit method which is at least $O(h^4)$ accurate. The source function f plays an important role in approximation of the terms u_{xxxy} and u_{xyyy} in (11). However, we have used only a discrete approximation for these terms. First we have substituted f_{xx} and f_{yy} for the terms u_{xxxy} and u_{xyyy} respectively. Thus we have obtained our compact exponential finite difference method

$$u_{i+1,j+1} - u_{i+1,j-1} - u_{i-1,j+1} + u_{i-1,j-1} = 4h^2 f_{ij} e^{\frac{h^2(f_{xx}+f_{yy})_{ij}}{6f_{ij}}} \tag{13}$$

where $u_{i+1,j+1}$ is an approximate value of the theoretical value of $u(x_i + h, y_j + h)$,..etc. For the computational purpose reported in section 4, we have used second order finite difference approximations for the terms f_{xx} and f_{yy} i.e.

$$(f_{xx})_{ij} = \frac{f_{i+1,j} - 2f_{ij} + f_{i-1,j}}{h^2} \tag{14}$$
$$(f_{yy})_{ij} = \frac{f_{i,j+1} - 2f_{ij} + f_{i,j-1}}{h^2}$$

To compute initial values in (13), we have the following algorithm reported in [9],

$$u_{i,j} = u_{i,j-1} + u_{i-1,j} - u_{i-1,j-1} + \frac{h^2}{4}(f_{i,j} + f_{i,j-1} + f_{i-1,j} + f_{i-1,j-1}) \tag{15}$$

3. The Local truncation error , Convergence and Stability Analysis

In this section, we consider the error associated with the proposed method (12. Let $u(x,y)$ be the solution of problem (1) six times continuously differentiable in the domain $[0,a] \times [0,a]$. Let T_{ij} be the truncation error in the proposed difference method (12) at mesh point (i,j) which may be defined as in [12] and we can write as

$$\begin{aligned} T_{ij} &= u_{i+1,j+1} - u_{i+1,j-1} - u_{i-1,j+1} + u_{i-1,j-1} - 4h^2 f_{ij} e^{\frac{h^2(f_{xx}+f_{yy})_{ij}}{6f_{ij}}} \\ &= \frac{h^6}{90}(3\frac{\partial^6 u}{\partial x^5 \partial y} + 10\frac{\partial^6 u}{\partial x^3 \partial y^3} + 3\frac{\partial^6 u}{\partial x \partial y^5} - \frac{5}{f}(\frac{\partial^4 u}{\partial x^3 \partial y} + \frac{\partial^4 u}{\partial x \partial y^3})^2)_{ij} \end{aligned} \tag{16}$$

Thus the order of the method (12) is four. Let us define error equation for difference method (12) as

$$\begin{aligned} &\epsilon_{i+1,j+1} - \epsilon_{i+1,j-1} - \epsilon_{i-1,j+1} + \epsilon_{i-1,j-1} \\ &\qquad = 4h^2(f(x_i,y_j,u(x_i,y_j)) - f(x_i,y_j,u_{ij})) + O(h^6) \end{aligned}$$

where $\epsilon_{ij} = u(x_i,y_j) - u_{ij}$. Using mean value theorem we have

$$\epsilon_{i+1,j+1} + \epsilon_{i-1,j-1} = \epsilon_{i+1,j-1} + \epsilon_{i-1,j+1} + 4h^2 \epsilon_{ij}(\frac{\partial f}{\partial u})_{ij} + O(h^6) \tag{17}$$

Let us define E_{j+1}, the maximal error on the $(j+1)^{th}$ level i.e.

$$E_{j+1} = \max_i |\epsilon_{i,j+1}|$$

Thus from (16), we have

$$\max_i |E_{j+1} + E_{j-1}| \leq \max_i |E_{j+1} + E_{j-1}| + 4h^2|E_j|\left|(\frac{\partial f}{\partial u})_{ij}\right| + O(h^6)$$

Thus as $h \to 0$

$$4h^2|E_j|\left|(\frac{\partial f}{\partial u})_{ij}\right| = 0$$

Thus the method (12) converges. The numerical solution will contain roundoff error and let $\bar{\epsilon}_{ij}$ be the roundoff error defined as

$$\bar{u}_{ij} = u_{ij} + \bar{\epsilon}_{ij}$$

Since a difference equation governs the prorogation of errors, it is possible to write (16) as

$$\bar{\epsilon}_{i+1,j+1} + \bar{\epsilon}_{i-1,j-1} = \bar{\epsilon}_{i+1,j-1} + \bar{\epsilon}_{i-1,j+1} + 4h^2\bar{\epsilon}_{ij}(\frac{\partial f}{\partial u})_{ij} \tag{18}$$

For the difference equations with constant coefficients, the error may be expanded in a finite Fourier series [12]. Thus if the source function, f is linear and defined $\bar{\epsilon}_{m,n}$, as in [12].

$$\bar{\epsilon}_{m,n} = Ae^{(i\beta mh)}\xi^n$$

where β is real number and A is an arbitrary constant. So the equation (17) may be written as

$$\xi^2(e^{(2i\beta h)} - 1) - 4h^2e^{(i\beta h)}\frac{\partial f}{\partial u}\xi - (e^{(2i\beta h)} - 1) = 0 \tag{19}$$

To simplify the equation (18), we have

$$\xi^2 + 2h^2\csc(\beta h)\cot(\beta h)\frac{\partial f}{\partial u}\xi - 1 = 0 \tag{20}$$

$$\xi^2 - 2h^2\sec(\beta h)\frac{\partial f}{\partial u}\xi - 1 = 0$$

The number ξ is called the amplification factor of the difference method. The method is stable iff $|\xi| \leq 1$. If source function f has arguments x and y only, then we see that the method is stable for all β. On solving (19), we conclude that the difference method (12) is stable if $\frac{\partial f}{\partial u} > 0$ then $\beta \in (\frac{\pi}{2}, \frac{3\pi}{2})$ and if $\frac{\partial f}{\partial u} < 0$ then $\beta < \frac{\pi}{2}$.

4. Numerical experiment

To illustrate our method and demonstrate its computational efficiency, we will consider the examples discussed in [8, 10], in which the errors taken to be the maximum absolute error i.e.

$$MAU = \max_{2\leq i,j\leq N} |u(x_i,y_j) - u_{ij}|$$

We have used Newton-Raphson iteration method to compute the values in (14). All computations in the experiment were performed on MS Window 2007 professional operating system in the GNU FORTRAN environment version -99 compiler(2.95 of gcc) running on Intel Duo core 2.20 Ghz PC. The solutions are computed on $(N-1)^2$ nodes, in computation of initial value iterations continued until either maximum difference between two iterates is less than 10^{-9} or number of iterations reached 10^3.

Problem 1. Consider a nonlinear problem discussed in [10]which,when solving consists of

$$u_{xy} = e^{(2u)}$$

in the region $[0,1] \times [0,1]$ with the boundary conditions $u(x,0) = x/2 - \log(1 + e^x)$, $u(0,y) = y/2 - \log(1 + e^y)$, for which the analytical solution is found to be $u(x,y) = (x + y)/2 - \log(e^x + e^y)$. For sake of comparison, we have computed the solution by the method in [13]. We have presented MAU by the present method (12) and method in [13], for different values of N in Table 1.

Problem 2. Consider a linear problem discussed in [8]which,when solving consists of

$$u_{xy} = u$$

in the region $[0,2] \times [0,2]$ with the boundary conditions $u(x,0) = e^x$, $u(0,y) = e^y$, for which the analytical solution is found to be $u(x,y) = e^{(x+y)}$. We have computed MAU by the present method (12) and the method in [13]. The computed MAU for both methods, for different values of N presented in Table 2.

Table 1. Maximum absolute error in $u(x,y) = (x + y)/2 - log(e^x/2 + e^y/2)$ for problem 1.

MAU	N			
	4	8	16	32
(12)	.36221743(-3)	.53644180(-4)	.67353249(-5)	.15497208(-5)
[13]	.61531067(-1)	.74501038(-2)	.11291504(-2)	.37765503(-3)

Table 2. Maximum absolute error in $u(x,y) = e^{(x+y)}$ for problem 2 .

MAU	N					
	4	8	16	32	64	128
(12)	.24116898(0)	.31757355(-1)	.41389465(-2)	.52642822(-3)	.80108643(-4)	.53405762(-4)
[13]	.57629776(0)	.61531067(-1)	.74501038(-2)	.11291504(-2)	.37765503(-3)	.72517395(-2)

5. Conclusion

In general, each numerical method has its own advantages and disadvantages of use. The present method is therefore good for use under the initial conditions. The major disadvantage of this method is in computation of nonlinear initial values.Our fourth order exponential finite difference method seems competitive with other finite difference methods.The decision to use a certain difference method does not depend on the given order of the method but also its computational efficiency. The numerical results for problems show that method is computational efficient. Also it is observed from the results that method has higher accuracy i.e smaller concretization error.

In the present article a different form of a high order method has been derived on the basis of exponential function. We have studied the accuracy and theoretical aspect of a developed finite difference method for numerical solutions of the Goursat problem.The development of this exponential method will lead to a possibility to approximate higher order derivatives in term of the power of lower order derivatives of solutions, to raise the order and accuracy of the method. Work in this direction is in progress.

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