Investigation of the De Morgan identities in fuzzy set theory[1]

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Abstract: In this paper we formulate and prove statements about the fulfillment of the fuzzy generalization of the De Morgan identities in cases of different t-norms, t-conorms and negations. In the proofs of the statements implicitly we propose methods to give uncountable infinitely many other norms and negations for a given norm, for which the triplets fulfill: none of the identities, exactly one of them, or both of them.

Keywords: Fuzzy connectives and aggregation operators; De Morgan identities

1. Introduction

The well-known De Morgan identities take a fundamental part of the basic knowledge of set theory (and logic). It is unnecessary to mention that their use can be noticed in countless places. In crisp (traditional) set theory these identities are generally true. (This is the reason they are also called *De Morgan's laws.*)

In case of all $\mathcal{U}$ universe in crisp set theory the De Morgan identities are fulfilled for arbitrary $A,B \subseteq \mathcal{U}$, and have the following forms [1]:

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \qquad \text{(disjunctive De Morgan identity)}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B} \qquad \text{(conjunctive De Morgan identity)}$$

In crisp set theory a set X can be defined by its characteristic function $\chi_X : \mathcal{U} \mapsto \{0,1\}$ as follows [1]: $\forall x \in \mathcal{U}$:

$$\chi_X(x) = \begin{cases} 1 & \text{, if } x \in X \\ 0 & \text{, if } x \notin X \end{cases}$$

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Since two sets (X and Y) are equal in crisp case if and only if $\forall x \in \mathcal{U}\colon\ \chi_X(x) = \chi_Y(x)$, where χ_X and χ_Y are the characteristic functions of X and Y, respectively [1], the previous identities can be written in the subsequent forms for arbitrary $A,B \subseteq \mathcal{U}$:

$$\forall x \in \mathcal{U}\colon\ \chi_{\overline{A \cup B}}(x) = c_c(u_c(\chi_A(x),\chi_B(x))) = i_c(c_c(\chi_A(x)),c_c(\chi_B(x))) = \chi_{\overline{A} \cap \overline{B}}(x)$$

$$\forall x \in \mathcal{U}\colon\ \chi_{\overline{A \cap B}}(x) = c_c(i_c(\chi_A(x),\chi_B(x))) = u_c(c_c(\chi_A(x)),c_c(\chi_B(x))) = \chi_{\overline{A} \cup \overline{B}}(x)$$

where i_c, u_c and c_c denote crisp intersection operator, crisp union operator, and crisp complement function respectively, hence $i_c:\ \{0,1\} \times \{0,1\} \mapsto \{0,1\}$, $u_c:\ \{0,1\} \times \{0,1\} \mapsto \{0,1\}$ and $c_c\colon\ \{0,1\} \mapsto \{0,1\}$, furthermore:

$$i_c(x,y) = \begin{cases} 1 & \text{, if } x = y = 1 \\ 0 & \text{, otherwise} \end{cases}$$

$$u_c(x,y) = \begin{cases} 0 & \text{, if } x = y = 0 \\ 1 & \text{, otherwise} \end{cases}$$

$$c_c(x) = \begin{cases} 0 & \text{, if } x = 1 \\ 1 & \text{, if } x = 0 \end{cases}$$

Fuzzy set theory [2] is a generalization of crisp set theory. It generalizes the characteristic functions and the operations on sets as well.

In fuzzy set theory a set X can be defined by its membership function (generalized characteristic function) $\mu_X : \mathcal{U} \mapsto [0,1]$ [3]. Two fuzzy sets (X and Y) are equal if and only if $\forall x \in \mathcal{U}\colon\ \mu_X(x) = \mu_Y(x)$, where μ_X and μ_Y are the membership functions of X and Y, respectively [3].

The fuzzy generalizations of crisp intersection, union and complement are all of those functions whose forms are $t\colon\ [0,1] \times [0,1] \mapsto [0,1]$, $s\colon\ [0,1] \times [0,1] \mapsto [0,1]$, $n\colon\ [0,1] \mapsto [0,1]$ respectively, and fulfill the following axioms [3, 4, 5]:

- t-norm (fuzzy intersection):
  i. boundary condition: $\forall x \in [0,1]\colon\ t(x,1) = x$,
  ii. monotonicity: $\forall x,y_1,y_2 \in [0,1]\colon$ if $y_1 \leq y_2$ then $t(x,y_1) \leq t(x,y_2)$,
  iii. commutativity: $\forall x,y \in [0,1]\colon\ t(x,y) = t(y,x)$,
  iv. associativity: $\forall x,y,z \in [0,1]\colon\ t(t(x,y),z) = t(x,t(y,z))$,
- t-conorm (fuzzy union):
  i. boundary condition: $\forall x \in [0,1]\colon\ s(x,0) = x$,
  ii. monotonicity: $\forall x,y_1,y_2 \in [0,1]\colon$ if $y_1 \leq y_2$ then $s(x,y_1) \leq s(x,y_2)$,

iii. commutativity: $\forall x,y \in [0,1]:\ s(x,y) = s(y,x)$,

iv. associativity: $\forall x,y,z \in [0,1]:\ s(s(x,y),z) = s(x,s(y,z))$,

- negation (fuzzy complement):

i. boundary conditions: $n(0) = 1$ and $n(1) = 0$,

ii. monotonicity: $\forall x,y \in [0,1]:$ if $x \leq y$ then $n(x) \geq n(y)$.

Some examples for fuzzy t-norms, t-conorms and negations [3, 4]:

- t-norms:

standard (or *Zadeh*) t-norm: $\forall x,y \in [0,1]:\ t(x,y) = min(x,y)$

algebraic t-norm: $\forall x,y \in [0,1]:\ t(x,y) = xy$

Łukasiewicz t-norm: $\forall x,y \in [0,1]:\ t(x,y) = max(x+y-1,0)$

nilpotent minimum: $\forall x,y \in [0,1]:$

$$t(x,y) = \begin{cases} min(x,y) & \text{, if } x+y>1 \\ 0 & \text{, otherwise} \end{cases}$$

drastic t-norm: $\forall x,y \in [0,1]:$

$$t(x,y) = \begin{cases} x & \text{, if } y=1 \\ y & \text{, if } x=1 \\ 0 & \text{, otherwise} \end{cases}$$

- t-conorms:

standard (or *Zadeh*) t-conorm: $\forall x,y \in [0,1]:\ s(x,y) = max(x,y)$

algebraic t-conorm: $\forall x,y \in [0,1]:\ s(x,y) = x+y-xy$

Łukasiewicz t-conorm: $\forall x,y \in [0,1]:\ t(x,y) = min(x+y,1)$

nilpotent maximum: $\forall x,y \in [0,1]:$

$$s(x,y) = \begin{cases} max(x,y) & \text{, if } x+y<1 \\ 1 & \text{, otherwise} \end{cases}$$

drastic t-conorm: $\forall x,y \in [0,1]:$

$$s(x,y) = \begin{cases} x & \text{, if } y=0 \\ y & \text{, if } x=0 \\ 1 & \text{, otherwise} \end{cases}$$

- negations:

  standard (or *Zadeh*) negation: $\forall x \in [0,1]: \ n(x) = 1 - x$

  intuitionistic negation:

$$n(x) = \begin{cases} 1 & \text{, if } x = 0 \\ 0 & \text{, if } x \in (0,1] \end{cases}$$

  dual intuitionistic negation:

$$n(x) = \begin{cases} 1 & \text{, if } x \in [0,1) \\ 0 & \text{, if } x = 1 \end{cases}$$

(Apparently arbitrary t-norm, t-conorm and negation can be combined in a triplet.)

On the basis of the previous generalizations, for arbitrary fuzzy sets $A,B \subseteq \mathcal{U}$, the fuzzy generalization of the De Morgan identities are:

$$\forall x \in \mathcal{U}: \ \mu_{\overline{A \cup B}}(x) = n(s(\mu_A(x),\mu_B(x))) = t(n(\mu_A(x)),n(\mu_B(x))) = \mu_{\overline{A} \cap \overline{B}}(x)$$

$$\forall x \in \mathcal{U}: \ \mu_{\overline{A \cap B}}(x) = n(t(\mu_A(x),\mu_B(x))) = s(n(\mu_A(x)),n(\mu_B(x))) = \mu_{\overline{A} \cup \overline{B}}(x)$$

Since A and B are arbitrary fuzzy sets, $\mu_A(x)$ and $\mu_B(x)$ can take any value from the interval $[0,1]$ for all $x \in \mathcal{U}$. Thus the identities can be written in the following forms [3]:

$$\forall x,y \in [0,1]: \ n(s(x,y)) = t(n(x),n(y)) \qquad \text{(disjunctive De Morgan identity)}$$

$$\forall x,y \in [0,1]: \ n(t(x,y)) = s(n(x),n(y)) \qquad \text{(conjunctive De Morgan identity)}$$

Because of the fact that there are uncountable infinitely many ways for the definition of t-norm, t-conorm and negation triplet, in fuzzy set theory the situation about the fulfillment of the De Morgan identities is more complicated than in crisp case. It is far from being obvious that the identities are fulfilled for arbitrary triplets.

The case of *involutive* negations has already been studied in [6]. The main result can be formulated as follows:

If the negation is involutive ($\forall x \in [0,1]: \ n(x) = n^{-1}(x)$), then in case of an arbitrary t-norm and t-conorm pair, none or both of the De Morgan identities are fulfilled.

The proof is not so difficult:
Assume that $\forall x,y \in [0,1]: \ n(s(x,y)) = t(n(x),n(y))$, one of the identities is fulfilled.

Then $\forall x,y \in [0,1]: \; s(x,y) = n^{-1}(t(n(x),n(y)))$. Since $\forall x \in [0,1]: \; n(n(x)) = x$ (involutive property), $\forall x,y \in [0,1]: \; s(n(x),n(y)) = n^{-1}(t(n(n(x)),n(n(y)))) = n(t(x,y))$. We got the other identity, too.
(The proof is similar when it is assumed at the beginning that the latter identity is fulfilled.)

The aim of this article is to investigate the behavior of the identities in case of different t-norm, t-conorm and negation triplets, where the negation is not involutive.

In the next section statements about the fulfillment of these generalized identities will be formulated and proved. The fulfillment depends on the properties of the chosen t-norm, t-conorm and negation we apply. There are triplets which fulfill:

- none of the identities,
- exactly one of them,
- both of them.

In the third section an example will be given when exactly one of the identities is fulfilled. Finally, in the last section we summarize the paper and point at the unsolved problematic cases whose solution can be the object of further research.

2. Theorems

Definition 1. *A t-norm t is idempotent [3, 4, 5] if* $\forall x \in [0,1]: \; t(x,x) = x$. *Similarly, a t-conorm s is idempotent if* $\forall x \in [0,1]: \; s(x,x) = x$.

According to [7] the only idempotent t-norm is the standard t-norm. Similarly, the only idempotent t-conorm is the standard t-conorm.

Definition 2. *A t-norm t is positive [4] if* $\forall x,y \in (0,1]: \; t(x,y) > 0$. *Similarly, a t-conorm s is positive if* $\forall x,y \in [0,1): \; s(x,y) < 1$.

Definition 3. *A negation n is strict [4] if it is strictly monotonic and continuous in* $[0,1]$.

Theorem 1. *If the standard t-norm (t-conorm) is given, then no t-conorm (t-norm) and* strict *negation pair can be found, for which the triplet fulfills exactly one De Morgan identity.*

Proof. Let us devide the possibilities into two cases:

a, Consider the case of the given standard t-norm ($t(x,y) = min(x,y)$), when the t-conorm is also the standard one ($s(x,y) = max(x,y)$) and the negation is arbitrary. With the assumption $x \leq y$, $n(x) \geq n(y)$ stands, because of the monotonicity of the negation. Therefore we get the following equations:

$$n(s(x,y)) = n(max(x,y)) = n(y) = min(n(x),n(y)) = t(n(x),n(y))$$

Similarly with the assumption $x \geq y$.
Thus, we got $n(s(x,y)) = t(n(x),n(y))$, one De Morgan identity.
Similarly we can get the other identity, too.

b, In case of the given standard t-norm, when the t-conorm is not the standard one and the negation is arbitrary, but strict, i.e. strictly monotonic and continuous (therefore bijective [8]):

- $\exists a \in [0,1]: \ n(s(x,x)) \neq n(x) = t(n(x),n(x))$, otherwise s would be idempotent, hence s would be the standard t-conorm. Thus there exists $x,y \in [0,1]$: $n(s(x,y)) \neq t(n(x),n(y))$.
- $\exists x \in [0,1]: \ n(t(x,x)) = n(x) \neq s(n(x),n(x))$, otherwise s would be idempotent, hence s would be the standard t-conorm. Thus there exists $x,y \in [0,1]$: $n(t(x,y)) \neq s(n(x),n(y))$.

Therefore none of the identities are fulfilled.

Thus, neither of the De Morgan identities, or both of them are fulfilled for given standard t-norm, if the negation can only be strict. The proof is similar for given standard t-conorm. □

Theorem 2. *If the standard t-norm (t-conorm) is given, then uncountable infinitely many t-conorm (t-norm) and negation pairs can be found, for which the triplets fulfill exactly one De Morgan identity.*

Proof. Let $x^* \in (0,1)$ and

$$s(x,y) := \begin{cases} max(x,y) & \text{, if } x \in [0,x^*] \text{ or } y \in [0,x^*] \\ 1 & \text{, if } x,y \in (x^*,1] \end{cases}$$

(See Figure 1.)

The previous function is a t-conorm, because:

i. boundary condition: $s(x,0) = max(x,0) = x$,

ii. monotonicity: if $y_1 \leq y_2$ then:

- $s(x,y_1) = max(x,y_1) \leq max(x,y_2) = s(x,y_2)$, if $x \in [0,x^*]$ or $y_1,y_2 \in [0,x^*]$,
- $s(x,y_1) = max(x,y_1) \leq 1 = s(x,y_2)$, if $y_1 \in [0,x^*]$ and $x,y_2 \in (x^*,1]$,

- $s(x,y_1) = 1 \leq 1 = s(x,y_2)$,
  if $x,y_1,y_2 \in (x^*,1]$,

iii. commutativity:

- $s(x,y) = max(x,y) = max(y,x) = s(y,x)$,
  if $x \in [0,x^*]$ or $y \in [0,x^*]$,
- $s(x,y) = 1 = s(y,x)$,
  if $x,y \in (x^*,1]$,

iv. associativity:

- $s(s(x,y),z) = max(max(x,y),z) = max(x,y,z) = max(x,max(y,z)) = s(x,s(y,z))$,
  if $x,y \in [0,x^*]$ or $x,z \in [0,x^*]$ or $y,z \in [0,x^*]$
  (this includes the case when $x,y,z \in [0,x^*]$),
- $s(s(x,y),z) = max(1,z) = 1 = s(x,y) = s(x,max(y,z)) = s(x,s(y,z))$,
  if $z \in [0,x^*]$, but $x,y \in (x^*,1]$,
- $s(s(x,y),z) = s(max(x,y),z) = s(y,z) = 1 = max(x,1) = s(x,s(y,z))$,

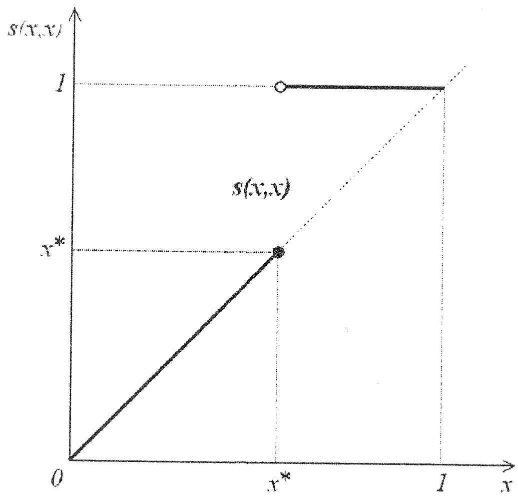

Figure 1: Main diagonal cross section of the enveloping cube of the graph of $s(x,y)$ (in Theorem 2)

if $x \in [0,x^*]$, but $y,z \in (x^*,1]$,

- $s(s(x,y),z) = s(max(x,y),z) = s(x,z) = s(x,max(y,z)) = s(x,s(y,z))$, if $y \in [0,x^*]$, but $x,z \in (x^*,1]$,
- $s(s(x,y),z) = s(1,z) = 1 = s(x,1) = s(x,s(y,z))$, if $x,y,z \in (x^*,1]$.

Let

$$n(x) := \begin{cases} 1 - \frac{x}{x^*} & \text{, if } x \in [0,x^*] \\ 0 & \text{, if } x \in (x^*,1] \end{cases}$$

(See Figure 2.)

The previous function is a negation, because:

i. boundary conditions: $n(0) = 1$ as well as $n(1) = 0$,

ii. monotonicity: $\forall x,y \in [0,1]$ where $x \leq y : n(x) \geq n(y)$.

Under the assumption $x \leq y$, $n(x) \geq n(y)$ stands, because of the monotonic decreasing property of the negation. Therefore we get the following equations:

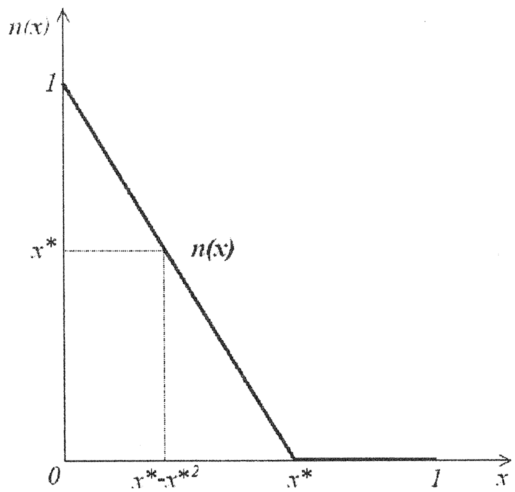

Figure 2: Graph of $n(x)$ (in Theorem 2 and 3)

- $n(s(x,y)) = n(max(x,y)) = n(y) = min(n(x),n(y)) = t(n(x),n(y))$, if $x \in [0,x^*]$ or $y \in [0,x^*]$
- $n(s(x,y)) = n(1) = 0 = n(y) = n(x) = min(n(x),n(y)) = t(n(x),n(y))$, if $x,y \in (x^*,1]$.

Similarly, with the assumption $x \geq y$. Thus $\forall x,y \in [0,1]: n(s(x,y)) = t(n(x),n(y))$, one De Morgan identity is fulfilled.

Since the negation is monotonic and decreasing:

$\forall x \in (0,x^* - x^{*2}): x^* = 1-(1-x^*) = 1 - \frac{x^*-x^{*2}}{x^*} = n(x^* - x^{*2}) < n(x) < n(0) = 1$, therefore

$\forall x \in (0,x^* - x^{*2}): s(n(x),n(x)) = 1 \neq n(x) = n(min(x,x)) = n(t(x,x))$, thus

$\exists x,y \in [0,1]: n(t(x,y)) \neq s(n(x),n(y))$, the other De Morgan identity is *not* fulfilled.

Since x^* can take continuum many values from the interval $(0,1)$, we obtained uncountable infinitely many solutions. The proof is similar for given standard t-conorm. □

Theorem 3. *If the drastic t-norm (t-conorm) is given, then uncountable infinitely many t-conorm (t-norm) and negation pairs can be found, for which the triplets fulfill exactly one De Morgan identity.*

Proof. Let $s(x,y)$ be the drastic t-conorm, $x^* \in (0,1)$ and

$$n(x) := \begin{cases} 1 - \frac{x}{x^*} & \text{if } x \in [0,x^*] \\ 0 & \text{if } x \in (x^*,1] \end{cases}$$

(See Figure 2.)

The previous function is a negation, as it was explained in Theorem 2. Then the following are hold true:

- $\forall x,y \in (0,1]: n(s(x,y)) = n(1) = 0 = t(n(x),n(y))$,
- $\forall x \in [0,1]: n(s(x,0)) = n(x) = t(n(x),1) = t(n(x),n(0))$.

Therefore $\forall x,y \in [0,1]: n(s(x,y)) = t(n(x),n(y))$, one De Morgan identity is indeed fulfilled.

However, $\forall y \in (0,1): n(t(x^*,y)) = n(0) = 1 \neq n(y) = s(0,n(y)) = s(n(x^*),n(y))$. As a result, $\exists x,y \in [0,1]: n(t(x,y)) \neq s(n(x),n(y))$, the other De Morgan identity is *not* fulfilled.

Since x^* can take continuum many values from the interval $(0,1)$, we got uncountable infinitely many solutions. The proof is similar for given drastic t-conorm. □

Definition 4. *If n is a negation, then $x \in [0,1]$ is a fix point [9] if $n(x) = x$.*

Definition 5. *If n is a negation, then n is expanding [9] if $\forall x \in [0,1]: t(n(x),n(n(x))) \leq x \leq s(n(x),n(n(x)))$.*

According to [9], a bijective negation n with fixed point x_0 is expanding if and only if there exists an involutive negation N such that $\forall x \in [0,1]$:

$$N(x) \leq n(x), \text{ if } x < x_0$$

$$N(x) = n(x), \text{ if } x = x_0$$

$$N(x) \geq n(x), \text{ if } x > x_0$$

Definition 6. *Let us say that this function is strictly expanding if:*

$$N(x) = n(x), \text{ iff } x \in \{0,x_0,1\}$$

(See Figure 3.)

Theorem 4. *If a t-norm (t-conorm) is given, for which $\exists y^* \in (0,1): t(y^*,y^*) \in (0,y^*)$ (in case of t-conorm: $\exists y^* \in (0,1): s(y^*,y^*) \in (y^*,1)$), then uncountable infinitely many t-conorm (t-norm) and negation pairs can be found, for which the triplets fulfill exactly one De Morgan identity.*

Proof. Let n be an arbitrary strictly expanding negation that has a fix point at y^*. If we prove that such a t-conorm can be found for n, so that this pair with the given t-norm fulfills exactly one De Morgan identity, we prove the statement, because it is clear, that there are uncountable infinitely many strictly expanding negations which have a fix point at y^*.
An example for generating continuum many piecewise linear strictly expanding negations n which have a fix point at y^*:

$$f_1(x) := x\frac{y^*-1}{y^*} + 1 \qquad f_2(x) := x\frac{y^*}{y^*-1} + \frac{y^*}{1-y^*}$$

$$n(x) := \begin{cases} n_1(x) = x^{\frac{f_1(\frac{y^*}{2})-1}{p\frac{y^*}{2}}} + 1 = x^{\frac{y^*-1}{py^*}} + 1 & \text{, if } x \in [0, \frac{y^*}{2}] \\ n_2(x) = x^{\frac{y^*-n_1(\frac{y^*}{2})}{\frac{y^*}{2}}} + (y^* - y^{*\frac{y^*-n_1(\frac{y^*}{2})}{\frac{y^*}{2}}}) = \ldots & \text{, if } x \in (\frac{y^*}{2}, y^*] \\ n_3(x) = x^{\frac{\frac{f_2(\frac{y^*+1}{2})}{p}-y^*}{\frac{y^*+1}{2}}} + (y^* - y^{*\frac{\frac{f_2(\frac{y^*+1}{2})}{p}-y^*}{\frac{y^*+1}{2}}}) = \ldots & \text{, if } x \in (y^*, \frac{y^*+1}{2}] \\ n_4(x) = x^{\frac{-n_3(\frac{y^*+1}{2})}{\frac{y^*+1}{2}}} + \frac{n_3(\frac{y^*+1}{2})}{\frac{y^*+1}{2}} = \ldots & \text{, if } x \in (\frac{y^*+1}{2}, 1] \end{cases}$$

where p is a parameter and $p > 1$ (so it can take continuum many values). (See Figure 4.)

Let $s(x,y) := n^{-1}(t(n(x),n(y)))$ (since n is invertible, we have n^{-1}), then $s(x,y)$ is a t-conorm, because:

i. boundary condition: $s(x,0) = n^{-1}(t(n(x),1)) = n^{-1}(n(x)) = x$

ii. monotonicity: if $y_1 \leq y_2$ then:

$n(y_1) \geq n(y_2)$, because n is monotonic decreasing,

$t(n(x),n(y_1)) \geq t(n(x),n(y_2))$, because t is monotonic increasing,

$s(x,y_1) = n^{-1}(t(n(x),n(y_1))) \leq n^{-1}(t(n(x),n(y_2))) = s(x,y_2)$, because n^{-1} is monotonic decreasing (since if it was not, i.e. $\exists x_1,x_2 \in [0,1]: x_1 < x_2$

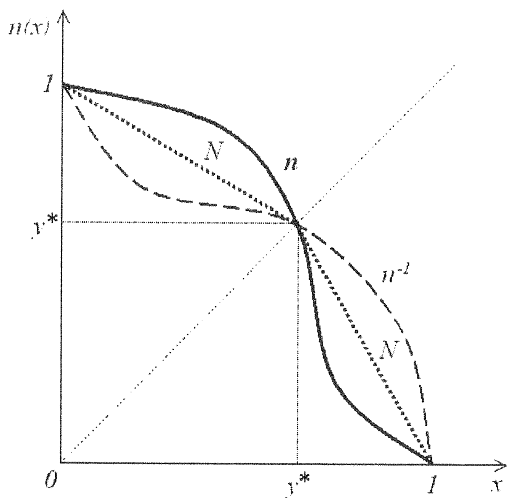

Figure 3: A strictly expanding negation (n) with its fix point (y^*) and its inverse

and $n^{-1}(x_1) \leq n^{-1}(x_2)$ stood, considering that n is monotonic decreasing, $x_1 = n(n^{-1}(x_1)) \geq n(n^{-1}(x_2)) = x_2$ would also be true)

iii. commutativity: $s(x,y) = n^{-1}(t(n(x),n(y))) = n^{-1}(t(n(y),n(x))) = s(y,x)$

iv. associativity:

$$s(s(x,y),z) = n^{-1}(t(n(n^{-1}(t(n(x),n(y)))),n(z))) = n^{-1}(t(t(n(x),n(y)),n(z))) = n^{-1}(t(n(x),t(n(y),n(z)))) = n^{-1}(t(n(x),n(n^{-1}(t(n(y),n(z)))))) = s(x,s(y,z))$$

One De Morgan identity is fulfilled, because $s(x,y) = n^{-1}(t(n(x),n(y)))$ if and only if $n(s(x,y)) = t(n(x),n(y))$. (Remember that n is invertible.)

Assume that the other identity is also fulfilled:

$\forall x,y \in [0,1]:\ n(t(x,y)) = s(n(x),n(y)) = n^{-1}(t(n(n(x)),n(n(y))))$, iff

$\forall x,y \in [0,1]:\ n(n(t(x,y))) = t(n(n(x)),n(n(y)))$, then

$n(n(t(y^*,y^*))) = t(n(n(y^*)),n(n(y^*)))$, then

$n(n(t(y^*,y^*))) = t(y^*,y^*)$, because $n(n(y^*)) = n(y^*) = y^*$.

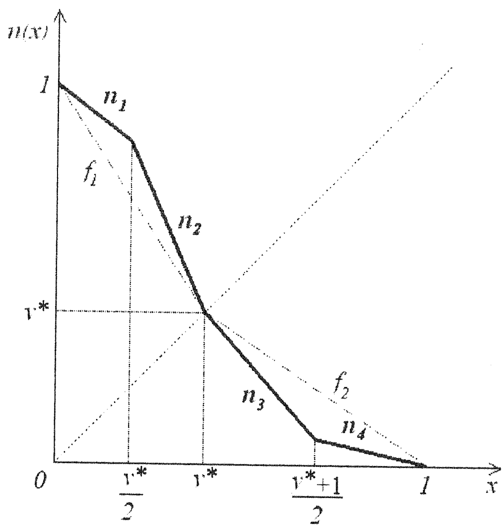

Figure 4: Graph of a piecewise linear strictly expanding negation

However, since n is strictly expanding:

$\forall x \in (0,y^*):\ n^{-1}(x) < N^{-1}(x)$, because

if $\exists x \in (0,y^*):\ n^{-1}(x) \geq N^{-1}(x)$ stood,

$\exists x \in (0,y^*):\ x = n(n^{-1}(x)) \leq n(N^{-1}(x)) < N(N^{-1}(x)) = x$, i.e.

$x < x$ would also be true, since $\forall x \in (0,y^*):\ N^{-1}(x) \in (y^*,1)$.

Thus:

$\forall x \in (0,y^*):\ n(x) > N(x) = N^{-1}(x) > n^{-1}(x)$, i.e.

$\forall x \in (0,y^*):\ n(x) > n^{-1}(x)$, hence

$\forall x \in (0,y^*):\ n(n(x)) < x$ (because n is strictly monotonic), thus

$n(n(t(y^*,y^*))) < t(y^*,y^*)$, since $t(y^*,y^*) \in (0,y^*)$ stands.

(See Figure 5.)

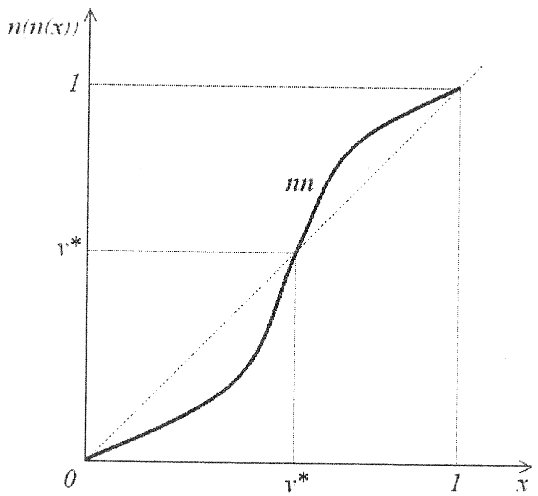

Figure 5: Graph of $n(n(x))$ (in Theorem 4)

That is a *contradiction.*

Therefore exactly one De Morgan identity is fulfilled. The proof is similar for given t-conorms. □

Corollary 1. *If a t-norm (t-conorm) is given that has at least one of the following properties:*

- *continuity (at least in one variable)*
- *strictly monotonicity*
- *positivity*

then uncountable infinitely many t-conorm (t-norm) and negation pairs can be found, for which the triplets fulfill exactly one De Morgan identity.

Proof. It can be easily seen that such a norm is either the standard one or the drastic one or for such a t-norm $\exists y^* \in (0,1): t(y^*,y^*) \in (0,y^*)$ (in case of t-conorm: $\exists y^* \in (0,1): s(y^*,y^*) \in (y^*,1)$). □

3. An example

Example 1. *Now let us see an example for a t-norm, t-conorm and negation triplet, which fulfills exactly one De Morgan identity.*

Let $\forall x,y \in [0,1]: t(x,y) := xy$ (the algebraic t-norm) and $\forall x \in [0,1]: n(x) := (x-1)^2$. Then $\forall x \in [0,1]: n^{-1}(x) = 1 - \sqrt{x}$, furthermore n is a negation, because:

i. boundary conditions: $n(0) = 1$ as well as $n(1) = 0$,

ii. monotonicity: if $x \leq y$ then $\forall x,y \in [0,1]: n(x) = (x-1)^2 = (1-x)^2 \geq (1-y)^2 = (y-1)^2 = n(y)$.

After this we can get the proper t-conorm as it is shown in Theorem 4:

$$s(x,y) = n^{-1}(t(n(x),n(y))) = 1 - \sqrt{(x-1)^2(y-1)^2} = 1 - |x-1||y-1| = 1 - (1-x)(1-y) = x + y - xy.$$

Apparently this is a t-conorm (accurately the algebraic t-conorm) and the triplet we got fulfills one De Morgan identity.

Assume that the other identity is also fulfilled:

$$\forall x,y \in [0,1]: s(n(x),n(y)) = n(t(x,y)), \text{ thus}$$

$$\forall x,y \in [0,1]: (x-1)^2 + (y-1)^2 - (x-1)^2(y-1)^2 = (xy-1)^2.$$

After multiplying and contracting we obtain:

$$\forall x,y \in [0,1]:\ x^2y^2 - x^2y - xy^2 + xy = 0.$$

However:

$$x^2y^2 - x^2y - xy^2 + xy = 0 \qquad \text{only if} \qquad x \in \{0,1\} \text{ or } y \in \{0,1\}.$$

Therefore we have a *contradiction.* Hence we showed a triplet, which fulfills exactly one De Morgan identity.

4. Conclusion

The meaning of the above proved statements can be summarized as follows:

I. For every given t-norm (t-conorm) can be found uncountable infinitely many t-conorm (t-norm) and negation pairs, for which the triplets:
   - do not fulfill any of the De Morgan identities. Namely, for a given standard norm we choose a non-standard norm and an arbitrary strict negation; for a given non-standard norm we choose a standard norm and an arbitrary strict negation. (See Theorem 1.)
   - fulfill both of the De Morgan identities. Namely, we choose an arbitrary involutive negation and a t-conorm, that is calculated as shown in Theorem 4. (See [6].)

II. For the standard, for the drastic, further for all such t-norms (t-conorms), where $\exists y^* \in (0,1):\ t(y^*,y^*) \in (0,y^*)$ (in case of t-conorm: $\exists y^* \in (0,1):\ s(y^*,y^*) \in (y^*,1)$) – these covers both the cases when the t-norm (t-conorm) is continuous, or strictly monotonic, or positive (see Corollary 1) – can be found uncountable infinitely many t-conorm (t-norm) and negation pairs, for which the triplets:
   - fulfill exactly one De Morgan identity. (See Theorem 2, 3 and 4.)

However, there are so-called problematic cases (see Figure 6), which are not covered by the statements. If the given t-norm (t-conorm) is not the standard one and is not the drastic one, further $\nexists y^* \in (0,1):\ t(y^*,y^*) \in (0,y^*)$ (in case of t-conorm: $\nexists y^* \in (0,1):\ s(y^*,y^*) \in (y^*,1)$), then the statements do not answer the question: can any t-conorm (t-norm) and negation pair be found, for which the triplet fulfills exactly one De Morgan identity?
(It is easy to prove, that there are uncountable infinitely many problematic cases.)

Further research may aim at finding an answer to the previous question.

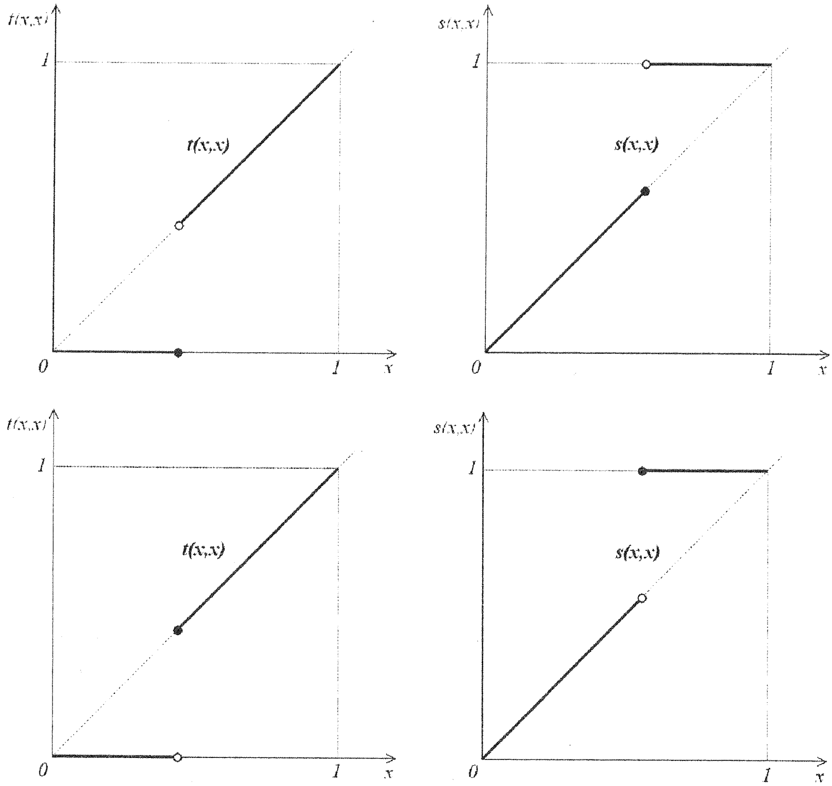

Figure 6: Main diagonal cross sections of the enveloping cubes of the graphs of the so-called problematic norms (in the Conclusion)

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