A control-oriented qLPV modeling framework

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Abstract: This paper proposes a framework for selecting affinely parametrized quasi Linear Parameter Varying (qLPV) model structures that facilitates solutions to specific control design tasks encountered in vehicle dynamics applications. Moreover it facilitates the selection of the scheduling variables and provides a framework to decide whether the controller performance can be improved by introducing some estimated parameters as scheduling variables, i.e., if some adaptive strategy is needed or not. The proposed scheme is an iterative process: in every step a suitable model transformation is applied to generate a finite element convex polytopic representation in order to obtain a qLPV model. Then the LMI feasibility of a robust control objective is verified, which is closely related to the original control task. This step provides a selection criterion that sorts out the suitable models from a finite set of model candidates generated by the iterative method.

Keywords: electric vehicle, nonlinear modelling, robust control, qLPV design

1. Introduction and motivation

In a control design problem a control law must be designed for a not entirely known system in order to reach given performance specifications. For a successful analysis and design, it is crucial to obtain a model that captures the essential behaviors of the system under consideration.

In modern control design the approximation of nonlinear models with linear models is often based on a qLPV description. This approach is based on the possibility of rewriting the plant in a form in which nonlinear terms can be hidden by using suitably defined scheduling variables by maintaining the linear structure of the model. An advantage of qLPV models is that in the entire operational interval nonlinear systems can be defined and a well-developed linear system theory to analyze and design nonlinear control system can be used.

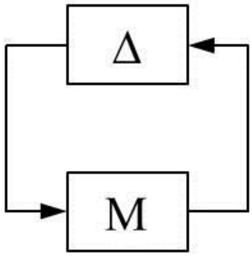

Figure 1. General feedback configuration

The models are augmented with performance specifications and uncertainties. Weighting functions are applied to the performance signals to meet performance specifications and guarantee a tradeoff between performances. The uncertainties are modelled by both unmodelled dynamics and parametric uncertainties. As a result of this construction a Linear Fractional Transformation (LFT) interconnection structure, which is the basis of control design, is achieved, see Figure 1.

These representations provide a particular structure to the LPV system, also known as a $\mathrm{M}-\Delta$ configuration, whereby the parameter-varying, uncertain or nonlinear terms are located in the diagonal Δ operator and the time invariant part is described by the operator M. An LFT based model set is widely considered to be the most general representation adopted in robust controller design.

It is apparent that there is a great amount of analogy between classical adaptive schemes and the qLPV design philosophy, see [1], [2]. The parameters that are estimated during operational time and which are used to tune the actual controller in an adaptive scheme play the same role as the scheduling variables in the qLPV context. From this latter perspective the difference consists in the acquisition of the scheduling variable, namely, in the adaptive case the values of the scheduling variable are not directly available by the measurement and need to be obtained by a specific estimation process based on the directly available data. This observation leads us to propose a unified view of both control design strategies cast in the qLPV design framework by extending the set of scheduling variables with parameters that might not be directly measured but estimated using a suitable designed procedure. The idea was tested through certain applications, see [3].

The solution to the LPV control synthesis problem is formulated as a parameter dependent LMI optimization problem, i.e. a convex problem for which efficient optimization techniques are available. This control structure is applicable whenever the value of parameter is available in real-time. The resulting controller is time-varying and smoothly scheduled by the values of the scheduling variables. Therefore qLPV models with Linear Matrix Inequalities (LMI), as the main design tool, seem to be the most efficient approach to achieve robust and non-conservative results.

Besides the weighting functions (performance and uncertainty weights) the model structure itself -- which is not unique -- influences decisively the success of the design and control quality. Concerning the latter the role of the uncertainty structure (modeling) is well known. It is less understood that in the LFT framework the choice of the scheduling variables affects the model in the same way as the uncertainties, moreover for a given model their choice is also non-unique, in general. The aim of this paper is to provide a systematic framework in which the search for a suitable model-concerning both uncertainty and scheduling variable structure- for a given control task can be performed.

1.1. The proposed modeling framework

The starting point is a (nominal) model

$$\begin{pmatrix} \dot{x} \\ z \\ y \end{pmatrix} = S(\theta, \pi) \begin{pmatrix} x \\ d \\ u \end{pmatrix} \quad (1)$$

where z is the performance vector, θ contains the measured variables, i.e., components/functions of y and some measured/estimated parameters, u is the control input, while d is the disturbance vector. The set of uncertain parameters is denoted by π.

The goal is to give a description of the type

$$\mathrm{S}(\theta, \pi) = \mathrm{S}_0 + \sum_{i \in I} \rho_i(\theta) \delta_j(\pi) \mathrm{S}_{i,j} \quad (2)$$

of the system which facilitates the control design task as much as possible where ρ_i will be the scheduling variables of the design while δ_j will catch the effect of the parametric uncertainties.

Robust control is handled based on the feedback connection depicted on Figure 0 and the associated well-posedness theorem, for details see [5]:

Theorem 1 *Let a subset* $\nabla \in \mathbb{C}^{k \times l}$ *and a matrix* $M \in \mathbb{R}^{l \times k}$ *be given. The following statements are equivalent:*

1. the feedback system on Figure 0 is well-posed, i.e., $det(I - \Delta M) \neq 0$ for all $\Delta \in \nabla$
2. there exist a symmetric matrix $P \in \mathbb{R}^{(k+l) \times (k+l)}$ such that

$$\begin{pmatrix} \mathrm{I} & \mathrm{M} \end{pmatrix} \mathrm{P} \begin{pmatrix} \mathrm{I} \\ \mathrm{M}^{\mathrm{T}} \end{pmatrix} < 0, \quad (3)$$

$$\begin{pmatrix} \Delta & \mathrm{I} \end{pmatrix} \mathrm{P} \begin{pmatrix} \Delta^{*} \\ \mathrm{I} \end{pmatrix} \geq 0 \quad \forall \Delta \in \nabla. \quad (4)$$

The constraint set in (4) is convex, however, it is usually not easily dealt with, since represents an infinity number of conditions. One way to overcome this difficulty is to approximate the exact set by a tractable one. By choosing appropriate inner/outer approximations one may develop computable lower/upper bounds for certain performances, e.g., stability margins.

As a possible solution, a uniformly and automatically executable Tensor Product (TP) model transformation method based on the recently developed Higher Order Singular Value Decomposition (HOSVD) concept has been proposed, see [7], [6]. The

TP model transformation offers uniform, tractable and readily executable numerical ways and creative manipulations to generate convex (polytopic) representations of LPV models upon which LMI-based design techniques are immediately executable. The result of the TP model transformation is a TP model that belongs to the class of polytopic models, where the parameter-dependent weightings of the vertex systems are one-dimensional functions of the elements of the parameter vector.

This form offers a relatively simple way to describe various convex hull generations in terms of matrix operations. The obtained structures are not unique, however the framework provides an efficient background to introduce a set of rules, heuristics and algorithms that provide us with a set of candidate model structures on which further analysis and final model selection can be carried out.

The selection criteria in the proposed framework can be tailored according to the given control task. The idea is to set an LMI feasibility problem related to a control-relevant task, e.g., robust stability with state feedback, robust performance with state feedback, etc., while solvability and the level of the achieved performances (if applicable) will provide the desired selection method.

The proposed framework facilitates the execution of the following program:

• build an qLPV model of the type (2),

• put the given model in the LFT form, e.g.,

$$\mathrm{S}(\theta,\pi) = \mathrm{S}_{1,1} + \mathrm{S}_{12}\Delta(\mathbb{I} - \mathrm{S}_{2,2}\Delta)^{-1}\mathrm{S}_{21} \tag{5}$$

where $\Delta = \begin{bmatrix} diag(\delta_i) & 0 \\ 0 & diag(\rho_j) \end{bmatrix}$ and $S_{i,j}$ are constant matrices,

• solve an LMI feasibility problem related to the control task,

• evaluate the results.

In order to make the method reliable the framework must provide efficient numerical techniques to perform each step. The aim of the paper is to propose such a framework.

The layout of the paper is the following: in Section 2 a brief description of the TP method is given. Section 3 gives details how the LMI problems suitable for the desired selection can be set. In Section 4 an example is provided to illustrate the proposed method. Finally, Section 5 contains some concluding remarks and future directions.

2. Tensor Product (TP) transformation for qLPV modeling

Tensor Product (TP) modeling, in broad sense, is an approximation technique where the approximating functions are in a tensor product form. The motivation is straightforward: one dimensional functions are much easier to calculate with, handle and visualize. A family of methods use tensor products of continuous univariate basis functions, e.g., non-uniform rational B-splines.

Consider a parameter-varying state-space model with input $\mathrm{u(t)}$, output $\mathrm{y(t)}$ and state vector $\mathrm{x(t)}$

$$\begin{bmatrix}\dot{\mathrm{x}}(\mathrm{t})\\ \mathrm{y}(\mathrm{t})\end{bmatrix} = \mathrm{S}(\mathrm{q}(\mathrm{t}))\begin{bmatrix}\mathrm{x}(\mathrm{t})\\ \mathrm{u}(\mathrm{t})\end{bmatrix} \tag{6}$$

with the parameter-varying system matrix

$$\mathrm{S}(\mathrm{q}(\mathrm{t})) = \begin{pmatrix}\mathrm{A}(\mathrm{q}(\mathrm{t})) & \mathrm{B}(\mathrm{q}(\mathrm{t}))\\ \mathrm{C}(\mathrm{q}(\mathrm{t})) & \mathrm{D}(\mathrm{q}(\mathrm{t}))\end{pmatrix}. \tag{7}$$

The time varying N-dimensional parameter vector $\mathbf{q}(t) \in \Omega$ is an element of the closed hypercube $\Omega = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_N, b_N] \subset \mathbb{R}^N$.

For practical reasons a finite element TP modeling is applied which uses a tensor defined by the values of $\mathbf{S}(\mathbf{q}(t))$ on a suitable discretization of Ω (usually a grid), i.e., a piecewise linear approximation of the multivariate map $\mathbf{S}(\mathbf{q}(t))$. Based on this data TP model transformation generates the HOSVD-based canonical form of LPV models [8], i.e.,

$$\begin{pmatrix}\dot{\mathrm{x}}(\mathrm{t})\\ \mathrm{y}(\mathrm{t})\end{pmatrix} = (\mathcal{S} \otimes_{\mathrm{n}=1}^{\mathrm{N}} \mathrm{w}_{\mathrm{n}}(\mathrm{q}_{\mathrm{n}}(\mathrm{t})))\begin{pmatrix}\mathrm{x}(\mathrm{t})\\ \mathrm{u}(\mathrm{t})\end{pmatrix}. \tag{8}$$

$\otimes_i$ denotes the i-mode tensor product as defined in [7]. For further details we refer to [6], [9].

This procedure extracts the unique structure of a given LPV model in the same sense as the HOSVD does for tensors and matrices, in a way such that:

- the number of LTI components are minimized;
- the weighting functions are univariate functions of the parameter vector in an orthonormed system for each parameter;
- the LTI systems are also in orthogonal position;
- the LTI systems and the weighting functions are ordered according to the higher-order singular values of the parameter vector.

Based on the higher-order singular values (that express the rank properties of the given model for each element of the parameter vector in L_2 norm), the TP model transformation offers a trade-off between the complexity of further design and the accuracy of the resulting TP model.

One of the advantages of the TP model transformation is that it can be executed uniformly (irrespective of whether the model is given in the form of analytical equations resulting from physical considerations, or as an outcome of soft computing based identification techniques such as neural networks or fuzzy logic based methods, or as a result of a black-box identification), without analytical interaction, within a reasonable amount of time. The obtained structure can be directly used for an LFT type modeling without any further preprocessing step.

Consider the map defined by the ordering $(i_1, \dots i_N) \to \mathbf{r}$ in the multi base number system defined by $(I_1, I_2, \dots, I_N)$. According to this indexing the weighting functions are denoted by

$$\mathrm{w_r(q(t)) = \prod_k w_{k,i_k}(q_k(t)) \in [0,1]},$$

where $w_{k,j}(q_k(t)) \in [0,1]$ is the j-th one variable weighting function defined on the k-th dimension of Ω, while the corresponding *vertex systems* are $\mathbf{S_r} = \mathbf{S}_{i_1, i_2, \dots, i_N}$. Using this index transformation one can write the TP model in the typical polytopic form:

$$\mathrm{S(q(t)) = \sum_{r=1}^{R} w_r(q(t)) S_r}. \tag{9}$$

Remark: Having $q(t) = [\theta(t), \pi]$ and the functions w_{n,i_n} are univariate the further splitting of the sum, i.e., $w_{\mathbf{r}}(q(t)) = \rho(\theta)_{\mathbf{r}} \delta(\pi)_{\mathbf{r}}$ is straightforward.

2.1. Multi-affine models

In many cases the convexity of the resulting TP model is required. The convex hull of $S(q)$ might not be polytopic, however for design purposes a finite, polytopic (outer) approximation is needed. Convexity is ensured by the following conditions:

$$\mathrm{\forall n \in [1, N], i, q_n(t) : w_{n,i}(q_n(t)) \in [0,1]}; \tag{10}$$

$$\mathrm{\forall n \in [1, N], q_n(t) : \sum_{i=1}^{I_n} w_{n,i}(q_n(t)) = 1}. \tag{11}$$

These conditions ensure that $\mathbf{S}(\mathbf{q}(t))$ is within the convex hull of the LTI vertex systems $\mathbf{S}_r$ for any $\mathbf{q}(t) \in \Omega$.

One of the main advantages of the TP model transformation is that we can find the convex representation via numerical matrix operations instead of analytical interactions. This approximation is highly nonunique and the TP approach provides a systematic approach in which different convex descriptions can be built. The TP model transformation was extended to generate different types of convex polytopic models, [10]. The generated convex hull of the polytopic models considerably influences the feasibility of the LMI-based design and the resulting performance level.

There are many ways to define the vertex systems and the type of the convex hull determined by the vertex system can be defined by the weighting functions. The applications of TP models specifies special requirements for the weighting functions.

For illustration purposes consider $\mathrm{S(q) = [q - q^2 \quad 2q]}$ where $\mathrm{q \in [-3,3]}$. In Figure 1 one can see the systems $\mathcal{S} = \mathrm{S(q)}$ (in blue). The dotted red lines depicts the directions given by the HOSVD while in green is depicted the smallest box that contains the convex hull $\tilde{\mathcal{S}}$ of $\mathcal{S}$. Another convex hull is depicted in magenta, that corresponds to a TP model. The corresponding weights are depicted in Figure 2.

It is worth noting that both the TP model transformation and the LMI-based control design methods are numerically executable one after the other, and this makes the resolution of a wide class of problems possible in a straightforward and tractable, numerical way.

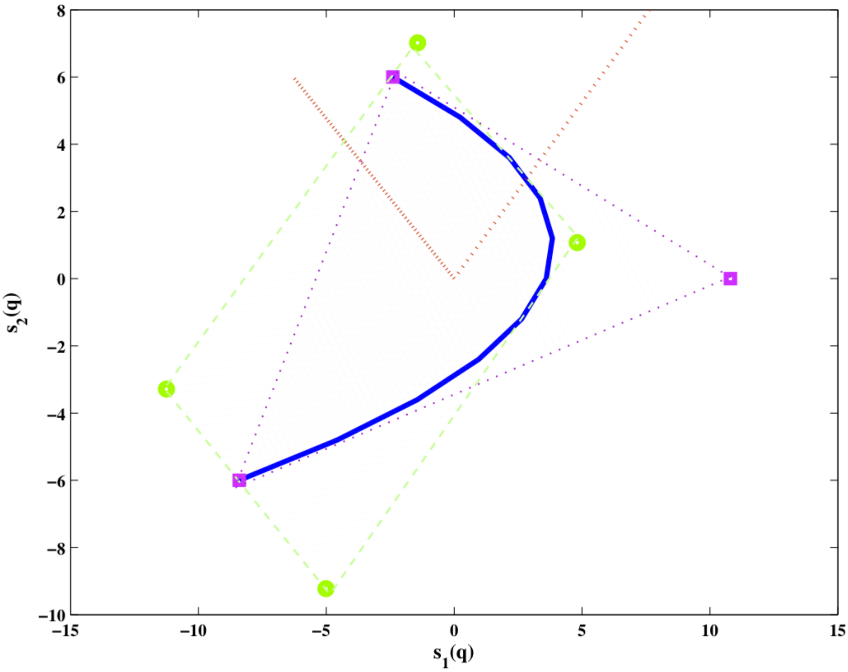

Figure 2. Different convex approximations

3. Setting LMI feasibility problems

Modern control design strategies strongly use LMI techniques. The variety of the control tasks affect the complexity of the resulting algorithms. For the purposes of this paper robust control objectives that lead to efficiently solvable LMI feasibility problems are to be selected.

Since output feedback control objectives often lead to non--convex bilinear matrix inequalities (BMI), which have computationally hard solution algorithms, this class of problems are not suitable candidates for a selection criteria. State feedback problems, however, usually lead to LMI feasibility problems, which can be solved more efficiently.

The easiest control objective is to stabilize the system. Let us recall that an LPV system is quadratically stable if $A(\rho)P + PA^T(\rho) < 0$ is fulfilled with a $P = P^T > 0$ matrix for all the parameters $\rho \in \mathcal{P}$. A necessary and sufficient condition for a system to be quadratically stable is that this condition holds for all the corner points of the parameter space, i.e., one can obtain a finite system of LMIs that must be fulfilled for $A(\rho)$ with a suitable positive definite matrix P, see [11], [12].

It follows that for the closed--loop system, i.e, for the matrices $A_c(\delta,\rho) = A(\delta,\rho) + B(\delta,\rho)K(\rho)$ the matrix inequality $A_c^T(\rho)P + PA_c(\rho) < 0$ must hold for suitable $K(\rho)$ and $P = P^T > 0$. By introducing the auxiliary variable $L(\rho) = PK(\rho)$, one can reduce the problem to a set of LMIs that must be solved at the corner points of the parameter space. This method makes possible to handle in a fairly straightforward way the parameter dependent feedback situation. However the method may lead to big LMI feasibility problems. This drawback can be eliminated by using relaxation techniques, e.g., for details see [13].

The drawback of using merely stabilizability as a selection criterion is that there is no direct information provided about the performance of the controller since there is no explicit performance criteria formulated in the problem. By doing simulations on relevant test scenarios, however, the different controllers, hence the different models, can be evaluated.

Fortunately, problems that contain meaningful performance specifications can be formulated in terms of LMI feasibility conditions. These problems can be set for systems of generalized LFT type:

$$\begin{pmatrix} \dot{x}(t) \\ z_u(t) \\ z_p(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} A & B_u & B_p & B \\ C_u & D_{uu} & D_{up} & E_u \\ C_p & D_{pu} & D_{pp} & E_p \\ C & F_u & F_p & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ w_u(t) \\ w_p(t) \\ u(t) \end{pmatrix}$$

$$\begin{pmatrix} w_u(t) \\ z_u(t) \end{pmatrix} \in \mathcal{S}(\Delta(t)) \subset \mathbb{R}^{m_u+k_u} \tag{12}$$

with the time-varying parameters satisfying $\Delta(t) \in \nabla$. It is assumed that $\mathcal{S}(\Delta)$ admits the explicit description $\mathcal{S}(\Delta) = Im(\mathbf{S}(\Delta))$ with a continuous matrix function $\mathbf{S}(\Delta)$ of full column rank. Furthermore, we suppose that (12) is well-posed, and that there exists a nominal value $\Delta_0 \in \nabla$ for which $Im\begin{pmatrix} 0 \\ I_{k_u} \end{pmatrix} \in \mathcal{S}(\Delta_0)$.

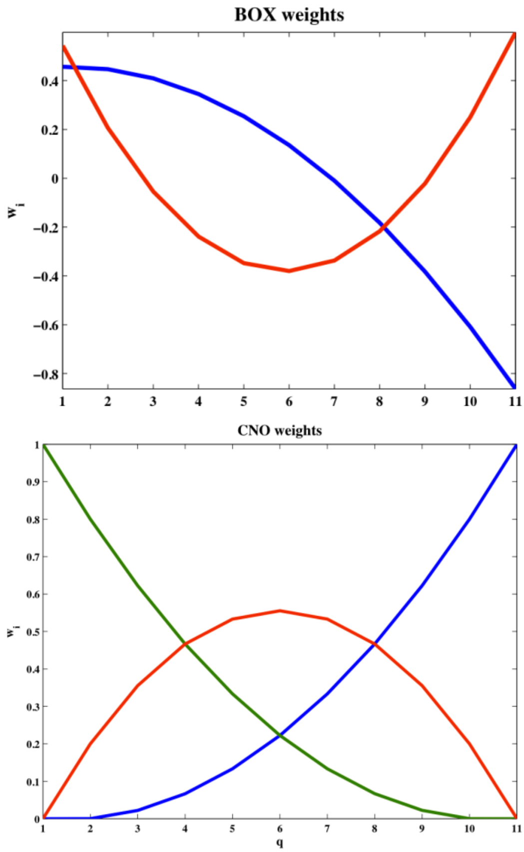

Figure 3. Weights for the different TP models

A -state-feedback or output feedback- controller is searched to fulfill a quadratic performance index:

$$\int_0^\infty \begin{bmatrix} w \\ z \end{bmatrix}^T \begin{bmatrix} Q_p & S_p \\ S_p^T & R_p \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix} \leq -\varepsilon ||w||^2,$$

e.g., for an $\mathbb{L}^2$--gain specification one has $Q_p = -\gamma^2 \mathbb{I}, S_p = 0$ and $R_p = \mathbb{I}$. For these problems the performance index γ is an indicator on the quality of the controller.

An output-feedback LPV controller for (12) is described as

$$\begin{pmatrix} \dot{x}_c(t) \\ u(t) \\ z_c(t) \end{pmatrix} = \begin{pmatrix} A_c & B_{c1} & B_{c2} \\ C_{c1} & D_{c11} & D_{c12} \\ C_{c2} & D_{c21} & D_{c22} \end{pmatrix} \begin{pmatrix} x_c(t) \\ y(t) \\ w_c(t) \end{pmatrix}$$

$$\begin{pmatrix} \mathrm{w_c(t)} \\ \mathrm{z_c(t)} \end{pmatrix} \in \mathcal{S}_\mathrm{c}(\Delta(\mathrm{t})) \subset \mathbb{R}^{\mathrm{m_c+k_c}} \tag{13}$$

and consists of an LTI system in which the on-line measured parameter $\Delta(t)$ enters via an implicit constraint imposed by $\mathcal{S}_c(\Delta)$. Here $\mathcal{S}_c(\Delta)$ is a subspace that depends continuously on $\Delta \in \nabla$ and that satisfies $Im\begin{pmatrix} 0 \\ I_{k_c} \end{pmatrix} \in \mathcal{S}_c(\Delta_0)$.

An LPV controller can be obtained by using the following result, for details see e.g. [14], [15], [16]:

Theorem 2 (LPV synthesis) *There exist a controller (13) such that closed-loop system is well-posed and stable if and only if there exist* X, Y, *multipliers* $P = \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix}$ *and* $\tilde{P} = \begin{pmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{pmatrix}$ *with* $P > 0$ *on* $\mathcal{S}(\Delta)$ *and* $\tilde{P} < 0$ *on* $\mathcal{S}(\Delta)^\perp$ *for all* $\Delta \in \nabla$ *that satisfy the matrix inequalities*

$$\begin{pmatrix} \mathrm{X} & \mathrm{I} \\ \mathrm{I} & \mathrm{Y} \end{pmatrix} \geq 0, \tag{14}$$

$$\Psi^\mathrm{T}\begin{pmatrix} * \\ * \\ * \end{pmatrix}^\mathrm{T}\begin{pmatrix} 0 & \mathrm{X} & 0 & 0 & 0 & 0 \\ \mathrm{X} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{Q} & \mathrm{S} & 0 & 0 \\ 0 & 0 & \mathrm{S^T} & \mathrm{R} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{Q_p} & \mathrm{S_p} \\ 0 & 0 & 0 & 0 & \mathrm{S_p^T} & \mathrm{R_p} \end{pmatrix}\begin{pmatrix} \mathrm{I} & 0 & 0 \\ \mathrm{A} & \mathrm{B_u} & \mathrm{B_p} \\ 0 & \mathrm{I} & 0 \\ \mathrm{C_u} & \mathrm{D_{uu}} & \mathrm{D_{up}} \\ 0 & 0 & \mathrm{I} \\ \mathrm{C_p} & \mathrm{D_{pu}} & \mathrm{D_{pp}} \end{pmatrix}\Psi < 0, \tag{15}$$

$$\Phi^\mathrm{T}\begin{pmatrix} * \\ * \\ * \end{pmatrix}^\mathrm{T}\begin{pmatrix} 0 & \mathrm{Y} & 0 & 0 & 0 & 0 \\ \mathrm{Y} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \widetilde{\mathrm{Q}} & \tilde{\mathrm{S}} & 0 & 0 \\ 0 & 0 & \tilde{\mathrm{S}}^\mathrm{T} & \widetilde{\mathrm{R}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \widetilde{\mathrm{Q}}_\mathrm{p} & \tilde{\mathrm{S}}_\mathrm{p} \\ 0 & 0 & 0 & 0 & \tilde{\mathrm{S}}_\mathrm{p}{}^\mathrm{T} & \widetilde{\mathrm{R}}_\mathrm{p} \end{pmatrix}\begin{pmatrix} -\mathrm{A^T} & -\mathrm{C_u^T} & -\mathrm{C_p^T} \\ \mathrm{I} & 0 & 0 \\ -\mathrm{B_u^T} & -\mathrm{D_{uu}^T} & -\mathrm{D_{pu}^T} \\ 0 & \mathrm{I} & 0 \\ -\mathrm{B_p^T} & -\mathrm{D_{up}^T} & -\mathrm{D_{pp}^T} \\ 0 & 0 & \mathrm{I} \end{pmatrix}\Phi > 0, \tag{16}$$

where $\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix} = \mathrm{Ker}\begin{pmatrix} \mathrm{B^T} & \mathrm{E_u^T} & \mathrm{E_p^T} \end{pmatrix}$ and $\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix} = \mathrm{Ker}(\mathrm{C} \quad \mathrm{F_u} \quad \mathrm{F_p})$.

This basic setting for the controller synthesis can be varied depending on the problem at hand and on the actual demands. The information on the change rate of the measured scheduling variables can be introduced through the slightly extended design equations derived in [17] and [18]. The details of the controller construction are fairly standard, hence, are omitted. Some details on the construction of controller scheduling variables, however, are relevant for our topic:

Relaxation: the LMI conditions on the scaling matrices P and $\widetilde{\mathrm{P}}$ must hold on an infinite set. In order to make the problem tractable a so called relaxation technique, i.e., sufficient conditions that must hold on a finite set, are needed. However, this might lead to a conservative design, hence we want to reduce the relaxation "gap" .

Having convex weighting functions a sufficient condition for the double summation: $\sum_{i,j} w_i w_j P_{ij} > 0$ is

$$\mathrm{X_{ij}} = \mathrm{X_{ji}^T}, \mathrm{X_{ii}} \le \mathrm{P_{ii}}, \mathrm{X_{ij}} + \mathrm{X_{ji}} \le \mathrm{P_{ij}} + \mathrm{P_{ji}}, \mathrm{j} \ne \mathrm{i},$$

$$\mathrm{Y} = [\, \mathrm{X_{ij}} \,] > 0.$$

A recursive version can be formulated for multi-convex TP summations: $\sum_{\mathbf{i,j}} w_{\mathbf{i}} w_{\mathbf{j}} P_{\mathbf{ij}} > 0$:

$$\mathrm{X_{ikjs}} = \mathrm{X_{jsik}^T}, \mathrm{X_{ikik}} \le \mathrm{P_{ikik}}, \mathrm{X_{ikjs}} + \mathrm{X_{jsik}} \le \mathrm{P_{ikjs}} + \mathrm{P_{jsik}},$$

$$\textstyle\sum_{\mathrm{i,j}} \mathrm{w_i w_j Y_{ij}} > 0, \quad \mathrm{Y_{ij}} = [\, \mathrm{X_{isjk}} \,], \quad (\mathrm{i,k}) \to \mathrm{i}.$$

Using the later technique stability can be proved even the stability domain is not convex, see [19]

Scheduling variables: the scheduling variables of the controller can be obtained applying the following procedure; perturb $\tilde{P}$, if required, to render it non-singular. Choose U such that its columns form an orthogonal basis of the image of $P - \tilde{P}^{-1}$. Define

$$\mathrm{M}(\Delta) = \mathrm{T^T}([\mathrm{U^T}(\mathrm{P} - \widetilde{\mathrm{P}}^{-1})\mathrm{U}]^{-1} - \mathrm{U^T S}(\Delta)(\mathrm{S}(\Delta)^{\mathrm{T}}\mathrm{PS}(\Delta))^{-1}\mathrm{S}(\Delta)^{\mathrm{T}}\mathrm{U})\mathrm{T}, \qquad (17)$$

where T is non-singular with

$$\mathrm{M}(\Delta_0) = \mathrm{diag}(\mathrm{M}_-, \mathrm{M}_+), \quad \mathrm{M}_- < 0, \quad \mathrm{M}_+ > 0.$$

Set $k_c = dim(M_+)$ and $m_c = dim(M_-)$. If $\mathbf{S}_c(\Delta)$ denotes the orthogonal projector onto the eigenspace of $M(\Delta)$ with respect to its positive eigenvalues, the continuous controller scheduling subspace of dimension k_c is given by $\mathcal{S}_c(\Delta) = Im(\mathbf{S}_c(\Delta))$.

Since expression (17) is quite complicated in general, by using a TP transform technique, one can obtain an affine parametrisation of the controllers scheduling block in terms of the original scheduling variables. Thus a more suitable expression that can be easily implemented is obtained.

4. Simulation example

In Figure 4. a two-degree-of-freedom quarter-car model is shown with body mass $\mathrm{m_s}$, unsprung mass $\mathrm{m_u}$, suspension stiffness $\mathrm{k_s}$, suspension damping $\mathrm{b_s}$ and tire stiffness k_t. The displacements of the sprung mass, the unsprung mass and their derivatives are q_1, q_2, $\dot{q}_1$ and $\dot{q}_2$, respectively. The system is excited by the road disturbance w and controlled by a force F.

Control performances of the suspension system are to keep sprung mass acceleration and suspension deflection small, and simultaneously limit the control force.

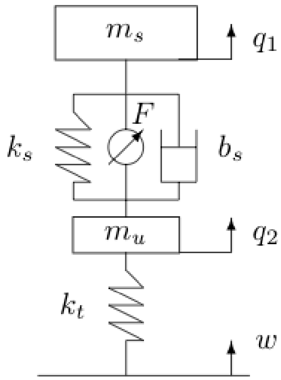

Figure 4. Quarter-car model

The vertical dynamics of the suspension system is formalized in the following way:

$$m_s\ddot{q}_1 = F_{ks} + F_{bs} - F, \tag{18}$$

$$m_u\ddot{q}_2 = -F_{ks} - F_{bs} - k_t(q_2 - w) + F, \tag{19}$$

where $F_{bs} = b_s^l\dot{d} - b_s^{sym}\dot{d}sgn(\dot{d}) + b_s^{nl}\sqrt{|\dot{d}|}sgn(\dot{d})$ is the suspension damping force and $F_{ks} = k_s^l d + k_s^{nl} d^3$ is the suspension spring force, with $d = q_2 - q_1$. The parts of the nonlinear suspension stiffness (k_s) are a linear coefficient k_s^l and a nonlinear coefficient k_s^{nl} while the nonlinear suspension damping b_s consists of a linear coefficient b_s^l and two nonlinear coefficients b_s^{nl} and b_s^{sym}, [20]. The measured outputs are d and $\dot{d}$.

The performance outputs are the passenger comfort (heave acceleration) ($z_a = \ddot{x}_1$), the suspension deflection ($z_s = x_s - x_u$) and the control input (z_u). The purpose of weighting functions W_{p_a}, W_{p_d}, and W_{p_u} in the closed-loop interconnection structure is to keep the heave acceleration, suspension deflection, wheel travel, and control input small over the desired frequency range. These weighting functions can be considered as penalty functions, i.e., weights should be large in a frequency range where small signals are desired and small where larger performance outputs can be tolerated.

The weighting functions for heave acceleration and suspension deflection are selected as $W_{p_a} = \phi_a W_{p_a,0}(s)$ and $W_{p_d} = \phi_d W_{p_d,0}(s)$, where parameter-dependent gains are applied to obtain trade-off between passenger comfort and road holding. A large gain ϕ_a and a small gain ϕ_d correspond to a design that emphasize passenger comfort. On the other hand, choosing ϕ_a small and ϕ_d large corresponds to a design that focuses on suspension deflection.

The LPV controller schedules on suspension deflection, and focuses on minimizing either the heave acceleration or suspension deflection response, depending on the magnitude of the vertical suspension deflection. In order to achieve the shift in focus from vertical acceleration to suspension deflection the weights associated with these signals are chosen to be parameter-dependent. In the mechanism two parameters are defined: c_1 and c_2. When the suspension deflection d is below c_1, the gain ϕ_a is selected to be constant and the gain ϕ_d is zero. When the deflection is between c_1 and c_2 the gains change linearly. When the value of the suspension deflection is greater than c_2, the gain ϕ_d is constant and the gain ϕ_a is zero, see Figure 5 for ϕ_d.

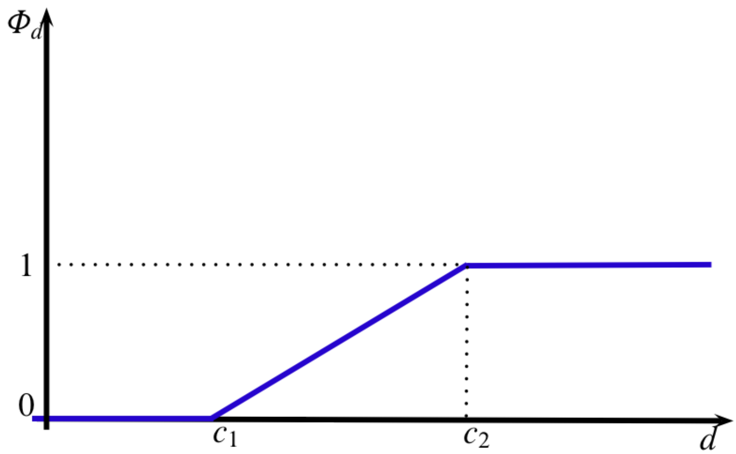

Figure 5. Performance gain

The parameters of the quarter-car model used in the simulations are given in Table 1. The control oriented qLPV model considers the nonlinearity of the generalized plant by selecting as scheduling parameters the measured outputs d and $\dot{d}$. Due to the structure of the dynamical equations the nonlinearities of the plant are cancelled out by a static term, i.e.,

$$u_c = k_s^{nl} d^3 - b_s^{sym}|\dot{d}| + b_s^{nl}\sqrt{|\dot{d}|}\,\mathrm{sgn}(\dot{d}) + \bar{u}_c.$$

Thus the generalized plant will contain only the nonlinearities introduced by the performance weights, with the control signal $\bar{u}_c$.

Table 1. Parameters of the quarter-car model

Symbols	Values	Unit	Description
m_s	290	kg	body mass
m_u	59	kg	unsprung mass
$b_{s,lin}$	700	N/m/s	lin. susp. damping
$b_{s,sym}$	100	N/m/s	nonlin. susp. damping
$b_{s,nl}$	200	$N/(m/s)^{1/2}$	nonlin. susp. damping
$k_{s,lin}$	16182	N/m	lin. susp. stiffness
$k_{s,nl}$	235000	N/m^3	nonlin. susp. damping
k_t	190000	N/m	tire stiffness

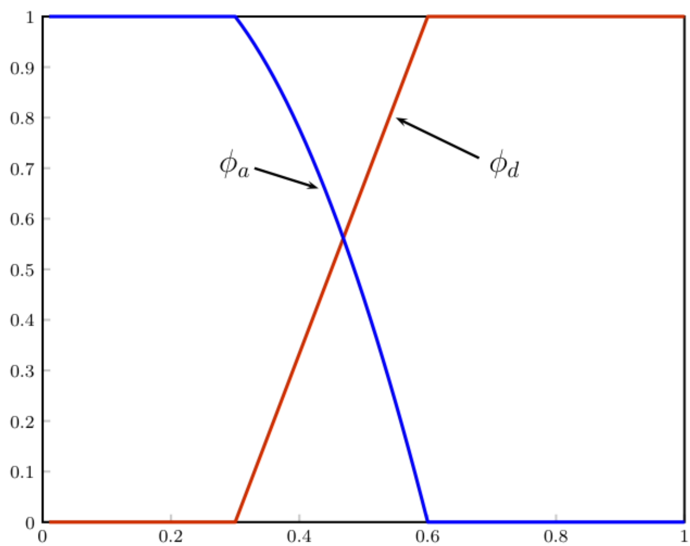

Figure 6. Gains of the performance weights: ϕ_a and ϕ_d

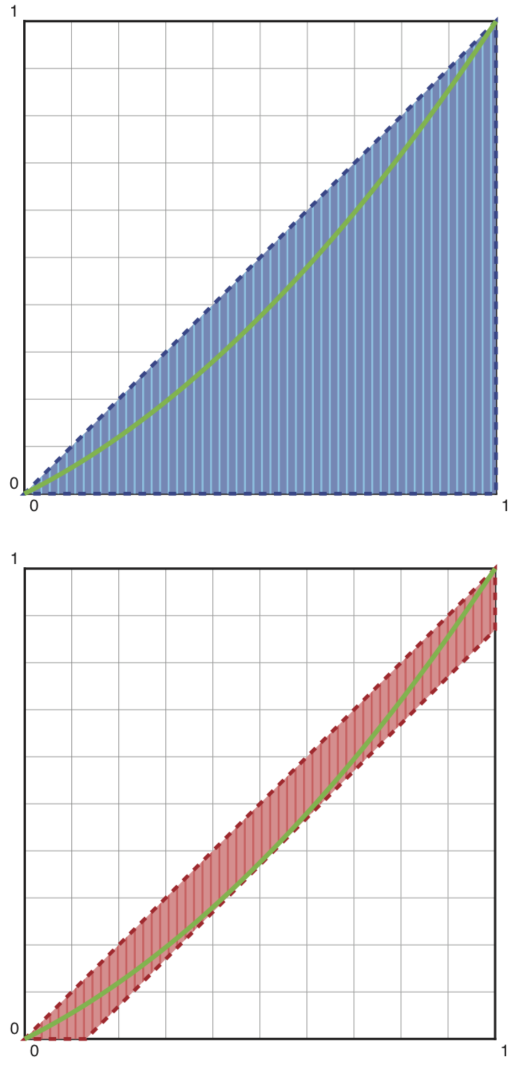

Figure 7. Convex relaxations for (ϕ_a, ϕ_d)

The weighting functions for heave acceleration, suspension deflection and control input are selected as

- $\mathrm{W_{p_a}} = (1 - 0.5\psi_\mathrm{d}(1 + \psi_\mathrm{d}))\mathrm{W_{p_a,0}}(\mathrm{s}),$
- $W_{p_d} = (1 - \psi_d)W_{p_d,0}(s),$
- $W_{p_u} = 1e - 4,$

with $W_{p_a,0}(s) = \frac{0.001s+0.3}{0.01s+1}$ and $W_{p_a,0}(s) = \frac{0.02s+6}{0.05s+1}$. The function $\psi_d(d)$ has the shape as in Figure 5, thus the qualitative shapes of the performance weights ϕ_a and ϕ_d are depicted o Figure 6. Note that the design guarantees stability for a convex region, i.e., one can tune the position of c_1 and c_2 according to the engineering needs. In the simulations these values were fixed to $c_1 = 5$ mm and $c2 = 10$ mm. Moreover, the tuning can be done in operational time. For an example for an application where such a tuning was exploited in order to achieve a desired behavior see [4].

For reference purposes two H_∞ controllers were designed where controller $H_{\infty,a}$ concentrates only on the heave acceleration while controller $H_{\infty,d}$ concentrates only on the minimization of the suspension deflection.

The convex relaxations used for diag($\phi_a \phi_d$) is depicted on Figure 7. Under the same conditions (weighting function, performance index) these tests have revealed that the value of the performance index that corresponds to the solution of the synthesis LMIs (15), (16) vary considerably depending on the choice made for the type of convex-hull. This result is in accordance with previous experiences obtained for stabilizing state feedback designs and indicates the influence of the convexification on the achievable performance in more complex settings, too.

Several qLPV controllers were design by using the tuning possibility of the LTI part of the controller. Two of them, qLPVa and qLPVb are included in the comparison in order to demonstrate the effects that can be achieved by such a tuning.

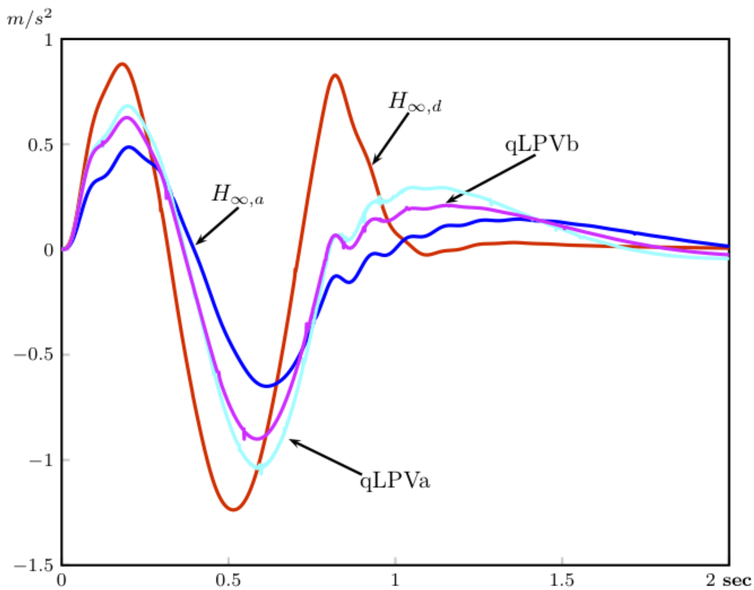

Figure 8. Achieved heave accelerations

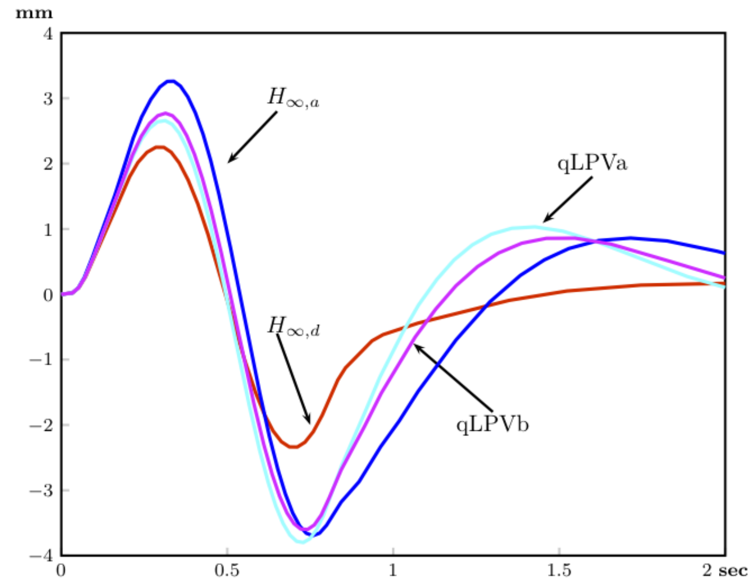

Figure 9. Achieved suspension deflections

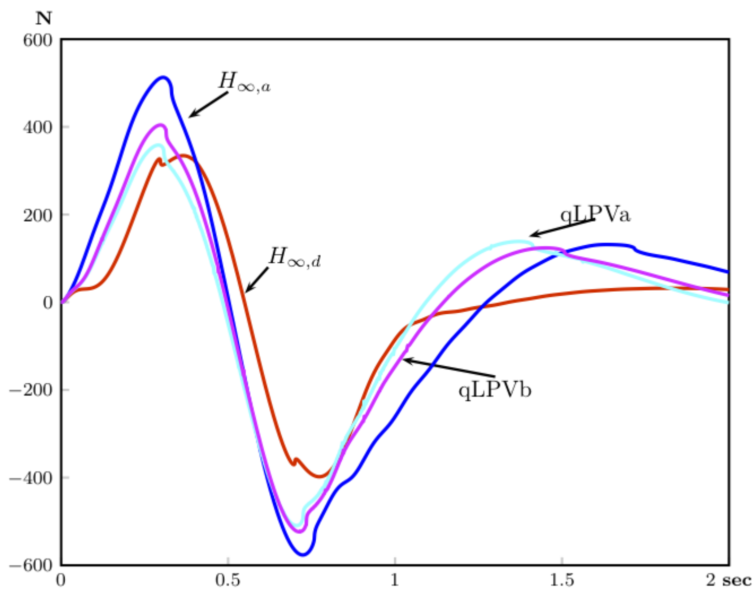

Figure 10. Control inputs of the designed controllers

The plots on Figure 8, 9 and 10 contain the achieved heave accelerations, the achieved suspension deflections and the applied control forces, respectively. The results reflects the achieved trade-off by the qLPV controllers between the conflicting multi-objective control criteria, i.e., road holding (suspension deflection) and passenger comfort (acceleration).

5. Conclusions

This paper has proposed methods to facilitate the design process of multi-objective qLPV robust control problems, often encountered in the design of vehicle systems, by efficient tuning possibilities. The proposed scheme is an iterative process in which a Tensor Product model transformation is applied to generate a finite element convex polytopic representation in order to obtain a quasi Linear Parameter Varying model. Then the LMI feasibility of a robust control objective is verified that is closely related by the original control task. This step provides a selection criterion that sorts out the suitable models from a finite set of model candidates generated by the TP method.

Since the choice of the most suitable convex relaxation has a great impact on the achievable performance, further research is done in order to provide algorithmic methods that facilitate the generation of different models by the TP technique. It is also a nontrivial question that for a given TP based model (9) how to derive the most suitable LFT description that fits the given control task.

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